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Rypdal, Kristoffer; Juul Rasmussen, Jens; Thomsen, K.

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Kristoffer Rypdal, Jens Juul Rasmussen,
Knud Thomsen

Risø National Laboratory, DK-4000 Roskilde, Denmark
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SIMILARITY SOLUTIONS OF THE CUBIC SCHRÖDINGER EQUATION AND THEIR ROLE IN THE DEVELOPMENT OF WAVE-COLLAPSE

Kristoffer Rypdal
IMR, University of Tromsø, N-9001 Tromsø, Norway

Jens Juul Rasmussen and Knud Thomsen
Association EURATOM - Risø National Laboratory
Physics Dept., Risø National Laboratory, DK-4000 Roskilde, Denmark

Abstract. Similarity transformations of the Cubic Schrödinger Equation (CSE) are investigated. The similarity transformations are used to remove the explicit time variation in the CSE and transform it into differential equations in the spatial variables only. Different methods for similarity reduction are employed and compared. The main purpose of this investigation is to study the significance of the similarity solutions in the evolution of a collapsing wave packet. Numerical solutions of the CSE in radial symmetry demonstrate that the similarity behaviour is local in space and time, and the similarity solutions are classified by invoking the concept of proper and improper solutions. The nature of the collapsing singularity is reexamined and finally, soliton solutions to the CSE are considered.

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I. INTRODUCTION

Certain nonlinear evolution equations of great physical significance exhibit solutions which develop a singularity within a finite time. In the mathematics literature this singular behaviour is most commonly called "blow-up", whereas in physics journals the same phenomenon is referred to as "wave-collapse". One analytical approach to these problems is the method of moments (Vlasov et al. 1971, Zakharov 1972), or the "virial theory" of collapse, as termed by some western authors (Goldman and Nicholson 1978). This method applies to equations which possess a certain hierarchy of conservation laws;

$$\partial_t \mathcal{N} + \nabla\cdot\vec{\mathcal{S}} = 0 \tag{1.1}$$

$$\partial_t \vec{\mathcal{S}} + \nabla\cdot\overleftrightarrow{T} = 0 \tag{1.2}$$

By defining an average position $\langle\vec{x}\rangle \equiv N^{-1} \int \mathcal{N}\, \vec{x}\, d\vec{x}$, and a mean square spatial width of the wave packet $\langle(\Delta\vec{x})^2\rangle \equiv N^{-1} \int \mathcal{N}(\vec{x} - \langle\vec{x}\rangle)^2 d\vec{x}$, where $N \equiv \int \mathcal{N}\, d\vec{x}$, one easily deduces from (1.1) and (1.2):

$$\partial_t^{\,2}\langle(\Delta\vec{x})^2\rangle = 2A \tag{1.3}$$

where $A \equiv (2/N) \int \mathrm{Tr}\, \overleftrightarrow{T}\, d\vec{x} - (\vec{S}/N)^2$, $\vec{S} \equiv \int \vec{\mathcal{S}}\, d\vec{x}$. A particularly simple case occurs if the next conservation law takes the form

$$\partial_t\, \mathrm{Tr}\, \overleftrightarrow{T} + \nabla\cdot\vec{Q} = 0 \tag{1.4}$$

from which it follows that A is a constant of motion, and eq. (1.3) can be integrated to yield

$$\langle(\Delta\vec{x})^2\rangle = At^2 + Bt + C \tag{1.5}$$

If $A < 0$, $\langle(\Delta\vec{x})^2\rangle \to 0$ in a finite time, and a singularity develops at the average position $\langle\vec{x}\rangle$.

Virial theory presupposes finite integrals. $N, S, \vec{A}$, i.e. for symmetric configurations the densities $\mathcal{N}, \vec{S}, \overleftrightarrow{T}$ must decay faster than r^{-D} as $r \to \infty$, where D is the num. : of spatial dimensions and $r \equiv |\vec{x}|$. It gives a sufficient criterion for singularity formation for localized (finite integrals) wave-forms. It does not, however, provide a necessary criterion because, as will be demonstrated in the following, a singularity may form locally even though $\langle(\Delta\vec{x})^2\rangle$ remains finite or even increases.

Another analytical approach is to search for self-similar substitutions. In most papers in the physics literature one tentatively writes the wave-field variable u on the form

$$u(\vec{x},t) = \frac{1}{(t_o-t)^\alpha}\,\phi(\vec{\eta}), \qquad \vec{\eta} \equiv \frac{\vec{x}}{(t_o-t)^\beta} \tag{1.6}$$

and tries to eliminate the time-dependence by the proper choice of α and β. The existence of localized solutions $\phi(\vec{\eta})$ is often taken as a necessary and sufficient criterion for self-similar collapse. As will be demonstrated in this paper, this criterion is neither necessary nor sufficient. It is not necessary because the evolution can be <u>locally</u> self-similar, i.e. the asymptotic behaviour of $\phi(\vec{\eta})$ for $\eta \to \infty$ is irrelevant. Furthermore it is not sufficient since there are cases where localized solutions do exist, which turn out to be unstable in the sense that infinitesimal perturbations either prevent the collapse, or bring it into a different self-similar mode as the collapse is approached. The stability analysis is performed by means of virial theory, since localized self-similar solutions must conform with the global predictions of this theory. The results of this analysis have also been verified numerically.

An important feature of self-similar behaviour is that many similarity substitutions are valid only if certain approximations are made in the original evolution equation. If the error becomes arbitrary small close to collapse, such a substitution may represent a real feature. This means that seeking exact solutions to "exact" evolution equations may rule out important classes of approximate solutions which become exact in the collapse limit.

An example from the current literature illustrates some of the points above. The problem is the existence of supersonic collapse as described by the so-called Zakharov equations (Zakharov 1972). Gol'tsman and Fraiman (1981) found a self-similar substitution for the full Zakharov system in two spatial dimensions. Localized solutions were found only for subsonic velocities, hence it was concluded that supersonic collapse does not exist. The authors did not take into account the fact that locally self-similar solutions may exist even in the absence of localized solutions of the equation for $\phi(\vec{\eta})$. Moreover, their subsonic solution was of the unstable type, and therefore had no physical relevance. Tskhakaya (1982) rediscovered this substitution, and by observing that it is valid for two space dimensions only, he concluded that Zakharov-collapse has a two-dimensional character. However, from the discovery of a similarity substitution alone, one can hardly draw any conclusions at all, and definitely not far-reaching ones like these (Rypdal et al. 1983). Zakharov and Shur (1981) applied the supersonic approximation prior to the self-similar substitution, and thus obtained a localized supersonic solution. Thereby they demonstrated the importance of selecting the proper approximations. However, these authors also seem to make a point of finding localized solutions, even though numerical experience indicates that this is quite irrelevant for a locally self-similar singularity formation (Budneva et al. 1975, Goldman et al. 1980).

In the present paper, we discuss in detail all the questions listed above, in the context of the Cubic Schrödinger Equation (CSE). This is perhaps the simplest model equation of physical interest, which exhibits all the features necessary for our discussion. It describes the nonlinear evolution of dispersive wave envelopes, and can be shown to possess a hierarchy of conservation laws (eqs. (1.1), (1.2) and (1.4)) which makes it tractable by means of virial theory. This means that for certain initial conditions, collapse is bound to occur. In addition to describe the time-evolution of a wave-envelope, the CSE also covers the stationary self-focusing of a light beam in a nonlinear, dispersive medium. The time-variable is then replaced by the axial coordinate. As there are two remaining spatial variables, this makes the two-dimensional case particularly interesting from an experimental viewpoint (Chiao et al. 1964, Garmire et al. 1966).

The report is organized as follows. In Sec. II we apply different similarity methods to reduce the number of variables in the Cubic Schrödinger Equation. Explicitly our aim is to remove the explicit time variation. We apply a general similarity transformation (II.1), a linear transformation group (II.2), and the Lie transformation group (II.3). In Sec. III we investigate if the self-similar solutions are localized or non-localized. The results of the numerical solutions of the CSE in two dimensions are discussed in Sec. IV: IV.1 describes the temporal evolution of amplitude and phase at $r = 0$, IV.2 the spatial structure of amplitude and phase, IV.3 the nature of the collapsing singularity, and finally IV.4 is concerned with the evolution of the global, localized similarity solution. In Sec. V we discuss the physical significance of the similarity solutions and we introduce the concepts of proper and improper solutions. In Sec. VI we consider the soliton solutions of the CSE and discuss their stability. Finally, Sec. VII contains our conclusions.

II. REDUCTION OF THE NUMBER OF VARIABLES IN THE CSE

Consider the Cubic Schrödinger Equation (CSE) in the following general form:

$$i\, u_t + \overleftrightarrow{p} : \nabla\nabla u + A|u|^2 u = 0 \tag{2.1}$$

where u describes the slowly varying envelope of a particular wave train and $\overleftrightarrow{p}$ is the dispersion tensor: $p_{ij} = \frac{1}{2}\partial^2\omega/\partial k_i \partial k_j$. As p_{ij} are assumed to be real, we can always perform a coordinate transformation that makes $\overleftrightarrow{p}$ diagonal. According to virial theory the existence of collapse can be guaranteed only if $\overleftrightarrow{p}$ is positively definite and $A > 0$.

Assuming this to be the case, we may scale the variables in the CSE to obtain:

$$i\, u_t + \nabla^2 u + |u|^2 u = 0 \tag{2.2}$$

which will be the starting point of our analysis.

II.1 A general similarity transformation

The aim of this section is to remove the explicit time variation from the CSE by performing similarity transformations. We apply a general self-similar transformation:

$$u(\vec{x},t) = \frac{\phi(\vec{\eta})}{f(t)} \exp[i\varphi(\vec{\eta},t)] \tag{2.3}$$

$$\vec{\eta} = \frac{\vec{x}}{h(t)} \tag{2.4}$$

to eq. (2.2):

$$2\Phi^2 + \frac{h'f}{f'h}\vec{\eta}\cdot\nabla\Phi^2 - \nabla\cdot\left(\frac{2f}{h^2f'}\Phi^2\nabla\varphi\right) = 0 \tag{2.5}$$

and

$$\nabla^2\phi + \frac{h^2}{f^2}\phi^3 + \left[hh'\vec{\eta}\cdot\nabla\varphi - h^2\varphi_t - (\nabla\varphi)^2\right]\phi = 0 \tag{2.6}$$

Here, the prime denotes differentiation and ∇ operates in $\vec{\eta}$-space. Demanding that Φ is not explicitly depending on t puts restrictions on f and h; which in principle make their explicit determination possible. We immediately see that $h = C\cdot f$ and since the constant C only corresponds to a trivial scaling factor we may put it equal to unity without loss of generality. Thus with $h \equiv f$ we find the following restrictions:

$$\frac{2}{ff'}\nabla\varphi \equiv \nabla\psi \qquad (\text{where } \psi = \psi(\vec{\eta})) \tag{2.7a}$$

$$ff'\vec{\eta}\cdot\nabla\varphi - f^2\varphi_t - (\nabla\varphi)^2 = g(\vec{\eta}) \tag{2.7b}$$

We "solve" (2.7a) for the phase:

$$\varphi(\vec{\eta},t) = \frac{ff'}{2}\psi(\vec{\eta}) + G(t) \tag{2.8}$$

where G(t) is an as yet unknown function. On using (2.8) in (2.5) and (2.7b) and using (2.7b) in (2.6) we arrive at the basic equations of the transformation:

$$(2-D)\phi^2 + \nabla\cdot\left[\vec{\eta}\phi^2 - \phi^2\nabla\psi\right] = 0 \tag{2.9}$$

$$\nabla^2\Phi + \Phi^3 + g(\vec{\eta})\Phi = 0 \tag{2.10}$$

$$g(\vec{\eta}) = \frac{(ff')^2}{2}\,(\vec{\eta}\cdot\nabla\psi - \tfrac{1}{2}(\nabla\psi)^2 - \psi) - \frac{f^3f''}{2}\,\psi - f^2\,G'(t) \tag{2.11}$$

where D is the number of spatial dimensions. In general $f(t)$ must satisfy the following equations in order that Φ does not depend explicitly on t:

$$(ff')^2 = 2b \tag{2.12a}$$

$$f^3f'' = -2a \tag{2.12b}$$

$$f^2\,G'(t) = \lambda \tag{2.12c}$$

where a, b and λ are constants. With f given (from (2.12a,b)), (2.12c) determines $G(t)$. The only solution for $f(t)$ satisfying both (2.12a) and (2.12b) is

$$f(t) = \left[2\sqrt{2a}\,(C\pm t)\right]^{\frac{1}{2}} \tag{2.13}$$

with $a = b > 0$. C is a constant which may be removed by a time translation, since Eq. (2.2) is invariant for time translations. We further note that since 'a' only is a trivial scaling factor it may be put equal to 1/8 without loss of generality. Jsing (2.13) in (2.12c) and (2.8) we determine the phase as

$$\varphi(\vec{\eta},t) = \pm\sqrt{a/2}\;\psi(\vec{\eta}) \pm \frac{\lambda}{2\sqrt{2a}}\,\ell n(C\pm t) + \varphi_o \tag{2.14}$$

where φ_o is a constant phase. The expression for $g(\vec{\eta})$ reads:

$$g(\vec{\eta}) = a\left[\vec{\eta}\cdot\nabla\psi - \tfrac{1}{2}(\nabla\psi)^2\right] - \lambda \tag{2.15}$$

$\Phi(\vec{\eta})$ and $\psi(\vec{\eta})$ may now be found by solving Eqs. (2.9) and (2.10) with $g(\vec{\eta})$ given in (2.15). The solution thus found, with the phase determined by (2.14), is an exact solution of CSE in any dimensions, and the equations (2.9),(2.10) with (2.14),(2.15) represents an exact reduction of eq. (2.2). The term exact should be understood in the broad sense that the transformation applies to the CSE without approximations. An approximate solution could for instance also be one where the restrictions (2.12) are satisfied approximately in a certain region in space and/or in a certain time domain. Such solutions will be discussed in a later section.

Unfortunately it is not possible to solve the equations (2.9), (2.10) and (2.15) in general. But we see that D = 2 constitutes a special case since Eq. (2.9) may be integrated directly for this case.

Assuming radial symmetry, eq. (..9) takes the form

$$\frac{dF}{d\eta} + \frac{1}{\eta} F = 0, \qquad F \equiv (\eta - \frac{d\psi}{d\eta})\phi^2 \tag{2.16}$$

which have the general solution

$$\frac{d\psi}{d\eta} = \eta + \frac{K}{\eta\phi^2} \tag{2.17}$$

where K is a constant of integration. If $\phi(0) \neq 0$, $d\psi/d\eta$

diverges at $\eta = 0$ for $K \neq 0$. However, such solutions are still of interest because similarity forms may be attained in limited spatial regions not including the origin. Hence, we first analyze the case $K \neq 0$.

By inserting eq. (2.17) into expression (2.11) for $g(\eta)$ eq. (2.10) takes the following form:

$$\frac{d^2\phi}{d\eta^2} + \frac{1}{\eta}\frac{d\phi}{d\eta} + \left\{\frac{a}{2}\left[\eta^2 - \frac{K^2}{\eta^2\phi^4}\right] - \lambda\right\}\phi + \phi^3 = 0 \tag{2.18}$$

This equation has the following asymptotic solution for large η:

$$\phi = \pm\sqrt{|K|}\left[\eta^{-1} + \frac{\lambda}{2a}\eta^{-3} + \mathcal{O}(\eta^{-4})\right] \tag{2.19}$$

Inserted into eq. (2.17), and integrated, this yields

$$\psi(\eta) = \psi_0 + \begin{cases} \eta^2 - (\lambda/a)\ln\eta + \mathcal{O}(\eta^{-1}) & \text{for } K > 0 \\ (\lambda/a)\ln\eta + \mathcal{O}(\eta^{-1}) & \text{for } K < 0 \end{cases} \tag{2.20a,b}$$

Inserting the expression for $K < 0$ in (2.20) into eq. (2.14) gives the following asymptotic expression for the phase:

$$\varphi(\eta,t) \sim \varphi_0 \pm \frac{\lambda}{2\sqrt{2a}}\ln\left[\eta^2(C\pm t)\right] + \mathcal{O}(\eta^{-1})$$

$$\sim \varphi_0 \pm \frac{\lambda}{\sqrt{2a}}\ln r + \mathcal{O}(\eta^{-1}) \tag{2.21}$$

by using (2.13).

Furthermore, we have from (2.19)

$$|u| = \frac{1}{f}\phi = \pm\frac{\sqrt{|K|}}{r} + \mathcal{O}(\eta^{-1}) \tag{2.22}$$

Note the time-independence of both amplitude and phase in the region where this asymptotic form is valid. The present solutions were first derived by Zakharov (1972).

We now proceed to the case K = 0, and eqs. (2.11) and (2.17) now yield

$$\psi(\eta) = \tfrac{1}{2}\eta^2 + \psi_o \tag{2.23}$$

$$g(\eta) = -f^3 f''\eta^2/4 - f^2\left[G'(t) + (f^2)''\,\psi_o/4\right] \tag{2.24}$$

Thus, in this case eq. (2.12) should be replaced by

$$f^3 f'' = -2a \tag{2.25a}$$

$$f^2\left[G'(t) + (f^2)''\,\psi_o/4\right] = \lambda \tag{2.25b}$$

Consequently, we have

$$G(t) = \lambda \int^t \frac{dt}{f^2} - (f^2)'\,\psi_o/4 \tag{2.26}$$

and eq. (2.8) then yields

$$\varphi(\eta,t) = (f^2)'\eta^2/8 + \lambda \int^t \frac{dt}{f^2} \tag{2.27}$$

By means of eqs. (2.24-25), eq. (2.10) reduces to

$$\frac{d^2\phi}{d\eta^2} + \frac{1}{\eta}\frac{d\phi}{d\eta} + \left(\frac{a}{2}\eta^2 - \lambda\right)\phi + \phi^3 = 0 \tag{2.28}$$

which is the special case of eq. (2.18) for K = 0. Asymptotic solutions decaying for large η are

$$\phi \sim \begin{cases} J_o(\frac{a}{2}\,\eta^2) & \text{for } a > 0 & \text{(a)} \\ K_o(\eta\sqrt{\lambda}) & \text{for } a = 0 & \text{(b)} \\ K_o(\frac{a}{2}\,\eta^2) & \text{for } a < 0 & \text{(c)} \end{cases}$$

(a) is oscillatory with amplitude decaying as η^{-1}, whereas (c) decays monotonically like $\eta^{-1}\exp(-a\eta^2/2)$. In sect. III we prove that for $a \neq 0$ no non-singular solution can exhibit an exponential decay as in (c), i.e. the solution corresponding to (c) is singular at $\eta = 0$. For $a = 0$, such a decaying solution does indeed exist, as will be discussed in sect. III. Note that even if $a \neq 0$, a solution exists which coincides approximately with the $a = 0$-solution for $a\eta^2/2 \ll \lambda$ even though it diverges for large η. Hence, this solution may characterize the waveform in a localized region around the origin even if $a \neq 0$. To find the time behaviour $f(t)$, we integrate (2.25a) to obtain

$$(f')^2 = \frac{2a}{f^2} + C_1 \tag{2.29}$$

We consider the three possibilities (i) $a > 0$, (ii) $a = 0$ and (iii) $a < 0$.

(i) $a > 0$:

For $C_1 = 0$ we reproduce the solution (2.13), whereas for $C_1 \neq 0$ we find

$$f(t) = \pm\left[C_1(t+C_2)^2 - 2a/C_1\right]^{\frac{1}{2}} \tag{2.30}$$

The integration constant C_2 demonstrates the invariance of the CSE with respect to time-translation. The phase corresponding to (2.30) follows from (2.27);

$$\varphi(\eta,t) = \frac{C_1}{4}(t+C_2)\eta^2 + \frac{\lambda}{2\sqrt{2a}} \ln \left| \frac{C_1(t+C_2) - \sqrt{2a}}{C_1(t+C_2) + \sqrt{2a}} \right| + \varphi_o \tag{2.31}$$

(ii) $a = 0$:

Now (2.29) is solvable only for $C_1 \geq 0$, yielding

$$f(t) = \pm(\sqrt{C_1}\, t + C_2) \tag{2.32}$$

$$\varphi(\eta,t) = \frac{C_1}{4}(t + C_2/\sqrt{C_1})\eta^2 - \frac{\lambda}{C_1 t + C_2\sqrt{C_1}} + \varphi_o \tag{2.33}$$

(iii) $a < 0$:

(2.29) has solutions only for $C_1 > 0$, yielding

$$f(t) = \pm\left[C_1(t+C_2)^2 + 2|a|/C_1\right]^{\frac{1}{2}} \tag{2.34}$$

$$\varphi(\eta,t) = \frac{C_1}{4}(t+C_2)\eta^2 + \frac{\lambda}{\sqrt{2|a|}} \arctan \frac{C_1(t+C_2)}{\sqrt{2|a|}} + \varphi_o \tag{2.35}$$

Note that in (2.34) $f(t)$ never becomes zero, i.e. the field function $u(r,t)$ exhibits no time singularity.

Possible solutions of the obtained ordinary differential equations (2.9-10) for $D \neq 2$ and (2.28) for $D = 2$ and their physical significance will be considered in the following sections. Some of the self-similar solutions considered here have also briefly been considered in other works (e.g. Talanov 1966; Ablowitz and Segur 1980; Redekopp 1980).

Finally we shall here briefly consider the special self-similar solutions having space independent amplitude, since for that case Eqs. (2.9-10) can be solved exactly. We thus take $\Phi(\vec{\eta}) \equiv \Phi_o$ (= const.) and (2.9) and (2.10) reduces to:

$$\nabla^2\psi = 2 \tag{2.36a}$$

and

$$g(\vec{\eta}) = -\Phi_o^2 \tag{2.36b}$$

A solution of (2.36a) is:

$$\psi = \frac{1}{D}\eta^2 + \frac{1}{D}\vec{C}_1\cdot\vec{\eta} + C_2 \tag{2.37}$$

By substituting (2.36b) and (2.37) into (2.11) we obtain the restrictions of f(t), however, it is shown that a consistent solution for f(t) can only be found for $\vec{C}_1 = \vec{0}$ and then (see appendix I):

$$f(t) = [2(C_3t+C_4)/D]^{D/2} \tag{2.38}$$

For the phase variation we obtain:

$$\varphi(\vec{\eta},t) = \frac{C_3}{2}\eta^2 + \frac{\Phi_o^2}{2C_3}\ln(C_3t+C_4) + \varphi_o \quad \text{for } D = 1$$

$$\varphi(\vec{\eta},t) = \frac{C_3}{2D}\left[\frac{2}{D}(C_3t+C_4)\right]^{D-1}\eta^2 - \frac{\Phi_o^2(D/2)^D}{C_3(D-1)}[C_3t-C_4]^{1-D} + \varphi_o \quad \text{for } D \neq 1 \tag{2.39}$$

where φ_o is a constant. Thus the solution of Eq. (2.2) for space-independent amplitude reads:

$$u = \frac{\phi_o}{\left[2(C_3t+C_4)/D\right]^{D/2}} \exp(i\varphi(\vec{\eta},t)) \tag{2.40}$$

with φ given by (2.39) and $\vec{\eta} = \vec{x}/f(t)$. We note that $2C_3/D$ is a simple scaling of t and may be put equal 1, whereas C_4 can be removed by a simple time translation. Doing that, the solution (2.40) reproduces the corresponding solutions given by Ablowitz and Segur (1979) and Redekopp (1980) for the one- and two-dimensional case. For the three-dimensional case our solution agrees with the solution of Lin and Strauss (1979) obtained for the CSE with a minus sign preceeding the dispersion term. From equation (2.40) we observe that decaying solutions to the CSE decay as

$$|u| = \phi_o t^{-D/2}$$

and the nonlinearity only appears in the phase. The solution in the one-dimensional case is associated with the decaying radiation part of the general inverse scattering solution of the CSE (Segur and Ablowitz 1976).

We emphasize that the constants appearing in the solutions for f(t) (e.g. Eq. (2.13,30,32,34 and 38)) may in principle be removed or changed by scaling or time translation. However, the freedom in choosing these constants may also be used to make the solutions of $u(\vec{x},t)$ fit prescribed boundary conditions or restrictions. If the solution $u(\vec{x},t)$ as given in Eq. (2.13) is demanded to be finite for all times $t \geq 0$, this requires that the constants are chosen such that $f(t) \neq 0$ for $t \geq 0$. On the other hand, if collapsing solutions are to be studied, we must require $f(t_o) = 0$, where t_o is the collapse time, and the constants must be chosen accordingly. Note in this connection that a self-similar solution determines its own initial- and boundary conditions. That is, if a given say initial condition should evolve according to a particular self-similar solution, then this initial condition must satisfy the reduced ordinary differential equations. It is thus obvious that obtaining a self-similar solution cannot generally be used to predict the occurrence of collapse, except for a very restricted class of initial conditions. Nevertheless, the forthcoming sections will demonstrate the significance of some of these similarity solutions in describing the collapse dynamics for a wide range of conditions.

II. 2 The linear transformation group

We shall here briefly describe how the explicit time variation is removed from the CSE (Eq. (2.2)) by applying a linear one parameter transformation group with the parameter p:

$$\hat{t} = p^k t; \; \hat{\vec{x}} = p^\ell \vec{x}; \; \hat{\Gamma} = p^m \Gamma; \; \hat{\varphi} = p^n \varphi \qquad (2.41)$$

where Γ and φ are modulus and phase of u, i.e. $u = \Gamma \exp(i\varphi)$, determined by the equations:

$$\nabla^2\Gamma + \Gamma^3 - \Gamma(\varphi_t + (\nabla\varphi)^2) = 0 \qquad (2.42)$$

$$\Gamma_t + 2\nabla\Gamma\cdot\nabla\varphi + \Gamma\nabla^2\varphi = 0 \tag{2.43}$$

These equations are invariant with respect to the transformation group (2.41) if and only if:

$$n = 0, \quad \frac{\ell}{k} = -\frac{m}{k} = \tfrac{1}{2} \tag{2.44}$$

The self-similar variables are now obtained as the invariants with respect to (2 41) (Bluman and Cole 1974):

$$\vec{\eta} = \frac{\vec{x}}{t^{\frac{1}{2}}}, \quad \Phi(\vec{\eta}) \equiv \frac{\Gamma(\vec{x},t)}{t^{-\frac{1}{2}}}, \quad \Psi(\vec{\eta}) = \frac{\varphi(\vec{x},t)}{t^{0}} \tag{2.45}$$

or

$$u = \frac{\Phi(\vec{\eta})}{t^{\frac{1}{2}}}\exp[i\Psi(\vec{\eta})] \tag{2.46}$$

By noting that the CSE is invariant for time translation the self-similar variables (2.45) are similar to the ones in (2.3) and (2.4) with $h(t) = f(t)$ determined by Eq. (2.13), except for the explicit time dependence of the phase. By inserting (2.45) into (2.42) and (2.43) we obtain a set of equations without the explicit time dependence, and in radial symmetry the equations reduce to a set of ordinary differential equations:

$$G\left(\begin{Bmatrix}\Phi\\ \Psi\end{Bmatrix}, \vec{\eta}\right) = 0 \tag{2.47}$$

A solution of (2.47) i.e. $\Phi(\vec{\eta})$ and $\Psi(\vec{\eta})$ then results in a space-time evolution determined by (2.45) and (2.46). This solution is the self-similar solution. A solution of the form $\Gamma + s(t)$, $\varphi + v(t)$ may also be self-similar if Γ and φ are self-similar and $s(t), v(t)$ are chosen to make the transformed equations invariant. Thus a wider class of similarity solutions can be obtained by performing the transformation:

$$\Gamma \rightarrow \Gamma + s(t), \qquad \varphi \rightarrow \varphi + v(t) \tag{2.48}$$

in Eqs. (2.42-2.43) prior to the application of the group (2.41). The invariance of the resulting equations (2.47) is preserved if, and only if, $s(t) \equiv 0$ and $v(t) \equiv \lambda \ln t$, where λ is an arbitrary real constant. Introducing that in (2.47) yields (in radial symmetry):

$$\frac{d^2\Phi}{d\eta^2} + \frac{D-1}{\eta}\frac{d\Phi}{d\eta} + (\tfrac{1}{2}\eta\theta - \theta^2 - \lambda)\Phi + \Phi^3 = 0 \tag{2.49}$$

$$\frac{d\theta}{d\eta} + \frac{D-1}{\eta}\theta - \tfrac{1}{2} = (\tfrac{1}{2}\eta - 2\theta)\frac{1}{\Phi}\frac{d\Phi}{d\eta} \tag{2.50}$$

where $\theta = d\Psi/d\eta$, and D is the number of spatial dimensions. The equations (2.49-2.50) are identical to Eqs. (2.9-2.10) (in radial symmetry) with f given by (2.13), the phase determined by (2.14) and $g(\eta)$ by (2.15).

To summarize, the only linear transformation group to which the CSE is invariant is given by (2.41) and (2.44). The self-similar solutions constructed from the invariants of this transformation group have the form of (2.46) or by applying the transformation (2.48) in addition:

$$u = \frac{\Phi(\eta)}{t^{\frac{1}{2}}} \exp[i\Psi(\eta) + \lambda \ln t]$$
$$\eta = \frac{r}{t^{\frac{1}{2}}} \tag{2.51}$$

where the dependent variables are determined from (2.49-2.50). No further solutions can be found by means of the linear transformation group. However. certain restrictions imposed on the solutions of (2.42-2.43) may reduce these to a set of equations, which are invariant with respect to other transformations. Normally, these restricted equations will be overdetermined, but in certain important cases (to find them may require an inventive mind) they are not As an example Tskhakaya (1982) found a transformation of the Zakharov equations which is valid for two space dimensions only, by imposing restrictions on the phase.

Let us illustrate this procedure briefly, i.e. we restrict the phase to be of the form:

$$\varphi = t^{\beta}\Psi(\vec{\eta}) + v(t), \quad \vec{\eta} = \frac{\vec{x}}{t^{\alpha}} \tag{2.52}$$

and select α, β and $\Psi(\vec{\eta})$ such that

$$\varphi_t + (\nabla\varphi)^2 = v_t \tag{2.53}$$

Inserting (2.52) into (2.53) we find:

$$\beta = 2\alpha-1 \quad \text{and}$$
$$(\nabla\Psi)^2 - \alpha\vec{\eta}\cdot\nabla\Psi + (2\alpha-1)\Psi = 0 \tag{2.54}$$

which has the solutions:

$$\Psi \equiv 0 \quad \text{or} \quad \Psi \equiv \eta^2/4 \tag{2.55}$$

The $\Psi \equiv 0$ solution yields from (2.42-2.43)

$$\nabla^2\Gamma - v_t\Gamma + \Gamma^3 = 0, \qquad \Gamma_t = 0 \tag{2.56}$$

Since Γ is time independent v_t must be chosen as a constant and the explicit time variation is eliminated. Formally the transformation group that leaves (2.56) invariant is the one corresponding to (2.41) with k arbitrary and $\ell = m = 0$. Discussions of the solutions of (2.56) will be presented in Sec. VI.

For the $\Psi = \eta^2/4$ solution Eqs. (2.42-2.43) reduce to

$$\nabla^2\Gamma - v_t\Gamma + \Gamma^3 = 0 \tag{2.57}$$

$$t\Gamma_t + \vec{\eta}\cdot\nabla\Gamma + \frac{D}{2}\,\Gamma = 0 \tag{2.58}$$

Invariance of this system with respect to (2.41) requires

$$\ell = -m \quad \text{and} \quad v(t) = \lambda t^{1-2\ell/k} \tag{2.59}$$

and in terms of the self-similar variables Eqs. (2.57-2.58) take the form:

$$\nabla^2\Phi - \lambda\Phi + \Phi^3 = 0 \tag{2.60}$$

$$\left(\frac{D}{2} - \frac{\ell}{k}\right)\Phi + \left(1 - \frac{\ell}{k}\right)\vec{\eta}\cdot\nabla\Phi = 0 \tag{2.61}$$

Obviously this set is overdetermined unless

$$\frac{\ell}{k} = 1 \quad \text{and} \quad D = 2 \tag{2.62}$$

which makes (2.61) trivially satisfied for any Φ. This transformation is seen to correspond to the transformation (2.3-2.4) for the two dimensional case with $h(t) = f(t) = (\sqrt{c_1}t+c_2)$, i.e. Eq. (2.32). The solution of (2.60) in radial symmetry (D = 2) has been shown to be localized (see Sec. III).

Concluding this subsection we have seen that a linear one parameter transformation group corresponds to the transformation Eqs. (2.3-2.4) with $f(t) = h(t) = (t+c)^{\beta}$. This is also seen from the way the similarity variables (the invariants with respect to the linear group) are constructed (Bluman and Cole 1974). Unless some restrictions are imposed on the solutions we find $\beta = \frac{1}{2}$.

II.3 The Lie transformation group

A broader spectrum of similarity solutions of the CSE than we have obtained in the previous sections should be obtainable by applying the infinitesimal transformation groups of Lie (e.g. Bluman and Cole 1974). Tajiri (1983) has applied this method to the general two dimensional CSE, and in order to reduce this equation, the transformation group was applied twice, each time the number of independent variables was reduced by one.

We shall not present a general reduction of the CSE by applying the infinitesimal group but only consider Eq. (2.2) in radial symmetry:

$$i\, u_t + u_{rr} + \frac{D-1}{r}\, u_r + |u|^2\, u = 0 \tag{2.63}$$

Thus the number of independent variables is reduced from D+1 to 2. This may also formally be done by applying a subgroup of the infinitesimal group that leaves (2.2) invariant. We shall now remove the explicit time variation in (2.63) by applying the infinitesimal one parameter (ε) Lie group:

$$\left.\begin{aligned} r^* &= r + \varepsilon R(r,t,u) + \sigma(\varepsilon^2) \\ t^* &= t + \varepsilon T(r,t,u) + \sigma(\varepsilon^2) \\ u^* &= u + \varepsilon U(r,t,u) + \sigma(\varepsilon^2) \end{aligned}\right\} \tag{2.64}$$

Since our main purpose is to investigate collapsing solutions we shall mainly consider the two- and three-dimensional cases. For D = 2 Eq. (2.63) corresponds to Eq. (4.12) of Tajiri (1983) (with his $\lambda \equiv r \equiv 1$), and by using his results we get the infinitesimals leaving (2.63) invariant with respect to the transformation (2.64):

$$\begin{aligned} R &= Ar + Brt \\ T &= 2At + Bt^2 + C \\ U &= \{iE + A + B(\tfrac{i}{4}\, r - t)\}u \end{aligned} \tag{2.65}$$

giving the following invariants, i.e. similarity variables ($\eta, \Phi(\eta)$ and $\Psi(\eta)$):

$$\eta = \frac{r}{\sqrt{|Q(t)|}}$$

$$u = \frac{\Phi(\eta)}{\sqrt{|Q(t)|}} \exp[i\varphi(\eta,t)] \tag{2.66}$$

where

$$\varphi = \frac{B}{4}\,\sigma\eta^2 t + \int \frac{E}{Q}\,dt + \Psi(\eta) \tag{2.67}$$

$$Q(t) = Bt^2 + 2At + C \tag{2.68}$$

$$\sigma = \begin{cases} +1 & \text{for } Q > 0 \\ -1 & \text{for } Q < 0 \end{cases} \tag{2.69}$$

and $\Phi(\eta)$, $\Psi(\eta)$ (note Tajiri's $f(\eta) \equiv \Phi(\eta)e^{i\Psi(\eta)}$) are determined by the two coupled ordinary differential equations:

$$\left(\frac{d}{d\eta} + \frac{1}{\eta}\right)\left(\frac{d\Psi}{d\eta}\,\Phi^2 - \frac{1}{2}\sigma A\Phi^2\eta\right) = 0 \tag{2.70}$$

$$\frac{d^2\Phi}{d\eta^2} + \frac{1}{\eta}\frac{d\Phi}{d\eta} + \Phi^3 + g(\eta)\Phi = 0 \tag{2.71}$$

with

$$g(\eta) = \sigma A\eta\,\frac{d\Psi}{d\eta} - \sigma E - \frac{BC}{4}\,\eta^2 - \left(\frac{d\Psi}{d\eta}\right)^2$$

Here the prime denotes differentiation with respect to η. Equation (2.70) is similar to Eq. (2.16) and its solution reads:

$$\frac{d\Psi}{d\eta} = \tfrac{1}{2}\sigma A\eta - \frac{K}{\eta\Phi^2} \tag{2.72}$$

where K is a constant. Introducing (2.72) in (2.71) we obtain:

$$\frac{d^2\phi}{d\eta^2} + \frac{1}{\eta}\frac{d\phi}{d\eta} + \phi^3 + \left\{\frac{1}{4}\,\eta^2(A^2-BC) - \frac{K^2}{\eta^2\phi^4} - \sigma E\right\}\phi = 0 \quad (2.73)$$

which resembles (2.18) and the discussion in connection with (2.18) also applies in connection with (2.73). We now take $K = 0$ in (2.72) and obtain:

$$\frac{d^2\phi}{d\eta^2} + \frac{1}{\eta}\frac{d\phi}{d\eta} + \phi^3 + (\frac{1}{4}\,\eta^2(A^2-BC) - \sigma E)\phi = 0 \qquad (2.74)$$

which resembles (2.28) with $a = \frac{1}{2}(A^2-BC)$ and $\lambda = \sigma E$. We note that $Q(t)$ becomes zero for $t = T_1$ and $t = T_2$ where

$$T_1 = -\frac{A}{B} + \frac{1}{B}(A^2-BC)^{\frac{1}{2}}, \quad T_2 = -\frac{A}{B} - \frac{1}{B}(A^2-BC)^{\frac{1}{2}} \quad (2.75)$$

Thus for $A^2 \geq BC$ $(a \geq 0)$ $Q(t)$ have real zeros and the solution (2.65) may describe collapse while for $A^2 < BC$ $(a < 0)$ no real zeros exist and the solution (2.65) cannot describe collapse. The special case $A^2 = BC$ $(a = 0)$ gives $T_1 = T_2$ and $Q(t) = B(T_2-t)^2$ and for $\sigma E > 0$ Eq. (2.74) has a localized solution as will be discussed in the next section. For $K = 0$ in (2.72) $\Psi = \frac{1}{4}\,\sigma A\eta^2 + \varphi_o$ and the phase reads

$$\varphi(\eta,t) = \frac{1}{4}\,\sigma\eta^2(A+Bt) + \int \frac{E}{Q}\,dt + \varphi_o \qquad (2.76)$$

From this expression it is easily seen that $\varphi(\eta,t)$ is equal to $\varphi(\eta,t)$ found in sec. II.1 for the three different cases:

$A^2 - BC \gtreqless 0$ $(a \gtreqless 0)$ Eqs. (2.31), (2.33) and (2.35).

We thus conclude that by applying the infinitesimal Lie group to the two dimensional CSE in radial symmetry we obtain the same similarity solutions as by using the transformation (2.3-2.4) Furthermore we do not obtain any "new" solutions.

We now apply (2.64) to the three dimensional CSE (Eq. (2.63) with D = 3) and find the infinitesimals leaving (2.63) invariant

$$R = Ar$$
$$T = 2At \tag{2.77}$$
$$U = (iE-A)u$$

resulting in the following invariants $(\eta, \Phi(\eta), \Psi(\eta))$

$$\eta = \frac{r}{|Q_1|^{\frac{1}{2}}}$$
$$u = \frac{\Phi(\eta)}{|Q_1|^{\frac{1}{2}}} \exp[i\varphi(\eta,t)] \tag{2.78}$$

where

$$\varphi(\eta,t) = \int \frac{E}{Q_1} dt + \Psi(\eta) \tag{2.79}$$

and

$$Q_1 = 2At + C \tag{2.80}$$

On substituting (2.78) with (2.79) and (2.80) into (2.63) (D = 3) we obtain a set of coupled ordinary differential equations for $\Phi(\eta)$ and $\Psi(\eta)$. These equations resembles Eqs. (2.9-2.11) in radial symmetry with f(t) given by Eq. (2.13) (i.e. $f(t) = \sqrt{|Q_1|}$) and $g(\eta)$ by Eq. (2.15). We further find that the expression

(2.79) for the phase resembles the phase given by Eq. (2.14). Therefore also for $D = 3$ we conclude that the application of the infinitesimal group (2.64) produce the same similarity solutions as the transformation (2.3-2.4) and no "new" solutions. We should emphasize, however, that this is not a general feature. For the one dimensional CSE, for instance, the transformation by the infinitesimal group indeed yield a much wider class of similarity solutions than obtained by the transformation (2.3-2.4) (see e.g. Tajiri (1983) Eqs. (4.2-4.5) and Johnson et al. (1979)).

III. LOCALIZED AND NON-LOCALIZED SOLUTIONS IN RADIAL SYMMETRY

A similarity solution will be called localized if $\phi(\eta) \to 0$ as $\eta \to \infty$ and the two conserved integrals

$$N \equiv R \int_0^\infty r^{D-1}\, dr\, |u|^2 \tag{3.1}$$

$$H \equiv R \int_0^\infty r^{D-1}\, dr(|u_r|^2 - \tfrac{1}{2}|u|^4) \tag{3.2}$$

are finite. Here $R = 2^{D-1}\pi$ for $D > 1$ and 2 for $D = 1$. By inserting the similarity substitution (2.3-4) and (2.8) we find

$$N = R \quad f^{D-2} \int_0^\infty \eta^{D-1}\, d\eta\, \phi^2 \tag{3.3}$$

$$H = R \quad \Big[f^{D-4} \int_0^\infty \eta^{D-1}\, d\eta(\phi_\eta{}^2 - \tfrac{1}{2}\phi^4)$$

$$+ \frac{1}{4} f^{D-2} f'^2 \int_0^\infty \eta^{D-1}\, d\eta\, \psi_\eta{}^2\, \phi^2\Big] \tag{3.4}$$

Since N and H are conserved, localized solutions are possible only if $D = 2$, $f' = \text{const.}$, and

$$\int_0^\infty \eta\, d\eta(\phi_\eta{}^2 - \tfrac{1}{2}\phi^4) = 0 \tag{3.5}$$

These conditions are satisfied by the $a = 0$ solution. Here f is given by eq. (2.32) for which $f' = \text{const.}$, and ϕ is given by the equation

$$\phi_{\eta\eta} + \eta^{-1}\phi_\eta + \phi^3 - \phi = 0 \tag{3.6}$$

In obtaining this equation, the constant λ in eq. (2.28) ($a = 0$) has been removed by a simple stretching, $\eta \rightarrow \eta/\sqrt{\lambda}$, $\phi \rightarrow \sqrt{\lambda}\,\phi$. This means that λ is absorbed into f, i.e. $f \rightarrow f/\sqrt{\lambda}$. Numerically it is easily demonstrated that (3.6) exhibits an infinite number of localized solutions ($\phi_\eta(0) = 0$, $\phi(\eta \rightarrow \infty) = 0$). In this countable set $\{\phi^{(j)}\}$, $j = 0,1,2,\ldots$, the solution $\phi^{(j)}(\eta)$ has j zeroes. For collapse purposes the most interesting solution is $\phi^{(o)}(\eta)$, which is monotonically decreasing. This solution is given by the dotted curve in fig. 3.

The other similarity solutions found in sect. II are all non-localized, since for these solutions f' is not constant. By this, we have proven that eq. (2.28) does not possess localized solutions for $a \neq 0$. In particular, this is the case for those solutions for which D = 2, $\psi(\eta) = \eta^2/2 + \psi_o$ and $\phi(\eta)$ satisfies eq. (2.28) for $a \neq 0$. From eq. (3.4) it is clear that these solutions are non-localized only if the integral $J \equiv \int_o^\infty \phi^2\eta^3 d\eta$ diverges. Hence, by means of the conservation laws for the partial differential equation (2.2) and a self-similar transformation, we have proved the following properties of the ordinary differential equation (2.28):

(i) For $a \neq 0$, the integral J is infinite for any non-singular solution $\phi(\eta)$.

(ii) For $a = 0$, the integral J is finite only if eq. (3.5) holds.

(ii) can also easily be proved directly from multiplying (3.6) by ϕ_η and integrating, but the authors know no simple way of proving (i) directly from (3.6).

IV. DISCUSSION OF NUMERICAL RESULTS FOR 2-DIMENSIONAL CSE

In sect. II we demonstrated that the 2-dimensional CSE reveals a number of similarity solutions which are specific for this dimensionality. As the main purpose of this paper is to assess the physical relevance of various similarity solutions, the 2-dimensional case will be the natural choice for a numerical study. The 1-dimensional case is well explored numerically, and analytically by means of the inverse scattering transform. 3-dimensional numerical studies have been performed by Budneva et al. (1975) and Goldman et al. (1980). The basic result of these studies was that the collapsing similarity solution corresponding to $f(t) = (t_o-t)^{\frac{1}{2}}$ is attained locally. In the following, a more complete study is made for D = 2 in radial symmetry. In particular, the spatial structure of the collapsing spike is analyzed for the first time.

We will first describe some of the general features of the collapson, as derived from the simulation. Some of the numerical details will be discussed later in this section.

Most of our numerical runs have started with a Gaussian wave-form with a standard deviation (spatial width) Δr and a maximum field amplitude u_o. An important parameter is $P = u_o^2/u_{thr}^2$, where $u_{thr} = \sqrt{2}/\Delta r$ is the virial theory threshold for collapse given by Goldman et al. (1980). A variation of Δr and u_o, such that $P = \frac{1}{2} u_o^2(\Delta r)^2$ is kept constant, has no qualitative consequence for the collapse dynamics, since the initial waveform can be kept invariant under a simple stretching transform which leaves the CSE invariant. Hence P is the only important parameter for our purpose, and we present results for P = 2. P = 20 and P = 100.

IV.1 Time evolution of $|u|$ and φ at $r = 0$

We test the hypothesis that the amplitude has an explosive growth of the form

$$|u(0,t)| \propto (t_o - t)^{-\beta} \tag{4.1}$$

as $t \to t_o$. We also test the logarithmic behaviour

$$\varphi(0,t) \propto \ln(t_o - t) \tag{4.2}$$

Behaviour of the form (4.1) and (4.2) follows from eqs. (2.30) and (2.31), respectively, in the limit $t \to t_o$, and also from eqs. (2.13) and (2.14).

For P = 2, the later part of the collapse is divided into two intervals where $\beta = 1/2$ and $\beta \approx 0.61$ respectively. The phase behaviour is shown in fig. 1 to be logarithmic in the $\beta = 1/2$-interval, in accordance with eq. (2.31).

For P = 20, $\beta = 1/2$ during the later stages of the collapse, the numerical scheme breaks down before the transition to higher β can take place. The scheme breaks down when the spatial width of the similarity spike becomes comparable with the numerical space step. Again a logarithmic phase behaviour is confirmed (fig. 1).

For P = 100, the $\beta = 1/2$-regime is entered just before the scheme breaks down.

Thus, the evolution agrees with the similarity forms (2.30-31) or (2.13-14) in a certain stage of the collapse evolution. When the spike-width becomes sufficiently small, β undergoes an adiabatic transition to larger values. The spike width, for which this takes place, decreases for increasing P.

IV.2 Spatial structure of $|u|$ and φ for different t

Fig. 2 gives an example of the space-time evolution of u and φ for P = 100. During the later stages a spike develops at r = 0. Qualitatively, the spatial profile can be divided into 3 distinct regions:

(I) The self-similar spike: In this region the amplitude profile approximately coincides with the function $f^{-1}\phi(r/f)$, where f(t) is determined from the time-evolution of the spike-maximum and $\phi(\eta)$ is the non-oscillatory localized solution of eq. (2.28) for a = 0. In the same region the phase is proportional with r^2, in agreement with eq. (2.33).

(II) The nonstationary transition region: As r increases the spatial profiles gradually depart from the similarity form until they enter a region where both amplitude and phase are almost stationary.

(III) The quasi-stationary region: On the time-scale of the spike development and collapse, the amplitude and phase in the outer regions of the collapson are almost stationary. The contraction of the wave-packet in this region occurs on the time-scale predicted by virial theory, and the virial theory collapse time turns out to be much later than the actual time for spike-collapse.

After this general description, we present some of the numerical results in more detail. If our similarity hypothesis were true, we could express the amplitude profile as $|u(r,t)| = f^{-1}\phi^{(0)}(r/f)$ where $\phi^{(0)}$ is the nonoscillatory, localized solution to equation (3.6).

Thus, in region (I) the expression

$$\tilde{\phi}(\eta;t) \equiv f\,|u(f\eta,t)| \qquad (4.3)$$

should be approximately equal to $\phi(\eta)$ throughout the entire region. In Fig. 3 $\phi(\eta)$ and $\tilde{\phi}(\eta;t)$ is plotted for different t for P = 2. The curve for t = 4.0 corresponds to the end of the β = 1/2-stage. It is seen that the amplitude profile deviates considerably from $\phi(\eta)$, even inside the peak halfwidth. The same type of behaviour is observed in the β = 1/2-stage for both P = 20 and P = 100. For later times, however, as the higher β (= 0.61) develops, the fit to $\phi(\eta)$ becomes progressively better. The spatial region where $\tilde{\phi}(\eta;t)$ progressively departs from $\phi(\eta)$, as η increases into the exponential tail of $\phi(\eta)$, corresponds to the nonstationary transition region (II). It is quite easily seen that the similarity region (I) cannot match directly onto the quasistationary region at a fixed $r = r_o$. If such a matching did exist at r_o, the corresponding $\eta_o = r_o/f$

would go to infinity as collapse is approached. The asymptotic behaviour of $\phi(\eta) \sim (\pi/(2\sqrt{\lambda}\ \eta))^{\frac{1}{2}} \exp(-\sqrt{\lambda}\ \eta)$ then requires that

$$f^{-\frac{1}{2}} \exp(-\sqrt{\lambda}\ r_o/f) = \text{const.} \tag{4.4}$$

which is obviously not the case when $f(t) \to 0$.

From our numerical runs we find a linear increase of the quasiparticle-number $N_{I+II} = 2\pi \int_{I+II} |u|^2\, r dr$ of the spike during the collapse development. This is due to the slower contraction of the field in region III, which provides a steady "flux of quasiparticles" into the spike region. During the earlier stages ($\beta = 1/2$) of the collapse the fractional increase of the quasiparticle number in the spike is considerable, allowing for a monotonic increase in the amplitude throughout the entire spike region (Fig. 4a). Closer to collapse ($\beta > 1/2$), $N_{I+II}(t)$ ($\approx N_{I+II}(t_o)$) remains nearly constant during the final collapse stage. Hence, the amplitude increase in the inner portion of the spike must be accompanied by a corresponding decrease in the outer portion in order to conserve quasiparticles (Fig. 4b). One might suspect that the transition from $\beta = 1/2$ to larger β is associated with this qualitative change of the spike dynamics. However, we are not able to prove this in analytical terms, as our knowledge about the dynamics in the transition region is very limited.

In Fig. 5 the spatial structure of the phase is displayed at various times. The inner portion, where φ is parabolic, corresponds to the similarity region. As there is no sharp boundary between region I and II it is difficult to give exact figures for the time-evolution of this boundary $r_o(t)$. Of course, quantitative criteria for determinations of $r_o(t)$ could be defined, but they would not be unique. Qualitatively, however, it is clearly seen from fig. 4b and fig. 5 that $r_o(t)$ is decreasing as $t \to t_o$. On the other hand, it is possible to see that the corresponding $\eta_o(t) \equiv r_o(t)/f(t)$ is slightly increasing. so that $r_o(t)$ decreases more slowly than does $f(t)$.

IV.3 Nature of the collapsing singularity

Zakharov et al. (1971) and Zakharov and Synakh (1975) argue theoretically that the final time evolution of the spike maximum corresponds to $f(t) = (t_o-t)^{2/3}$. We believe that their arguments are incorrect, but before demonstrating that, we will show that a self-similar evolution corresponding to $\beta > \frac{1}{2}$ is indeed an approximate solution of the CSE during late stages of a collapse. First, we note that our numerical solution demonstrates that $\tilde{\phi}(\eta;t)$ is not entirely time-independent; there is a certain "adiabatic" variation such that $\tilde{\phi}(\eta;t) \to \phi(\eta)$ as $t \to t_o$ in region I. For such a slow variation eqs. (2.9-11) might still be valid in the limit $t \to t_o$. In fact, a term of the form $(f/f')(\phi^2)_t$ should be added on the L.H.S. of eq. (2.9), and a slow time-variation shou_d be permitted in $\psi(\eta;t) = \eta^2/2 + (f/f')\psi_1(\eta;t)$, where $\psi_1(\eta;t)$ is of order unity for all t. Eq. (2.11) then takes the form

$$g(\vec{\eta};t) = \frac{f^3f'}{2}\,(\vec{\eta}\cdot\nabla\psi_1 - \tfrac{1}{2}(\nabla\psi_1)^2 - \psi_1) - \frac{f^3f''}{2}\,\psi - f^2G'(t) \tag{4.5}$$

The explicit t-dependence will vanish in the limit $t \to t_o$ provided

$$f/f' \to 0, \qquad f^3 f' \to 0, \qquad f^3 f'' \to 0 \tag{4.6}$$

and then

$$f^2(t)G(t) \to \lambda \quad \text{as } t \to t_o \tag{4.7}$$

where λ is a constant. As mentioned earlier $f(t)$ could be scaled such that $\lambda = 1$, and $f(t)$ should satisfy the equation

$$\phi^{(0)}(\eta=0) = f(t)u(0,t) \tag{4.8}$$

where the numerical solution of (3.6) yields $\phi^{(0)}(\eta=0) = 2.2062$. If we assume $f(t)$ has the form

$$f(t) = \alpha(t_o-t)^{\beta} \tag{4.9}$$

(4.6) requires that $\beta > \frac{1}{2}$. Applying the numerically obtained values for $u(0,t)$, we find from (4.8), and by means of (4.9) an excellent fit is obtained in the late stages for $P = 2$ if we choose $\alpha = 1.37$ and $\beta \doteq 0.61$. We can also check whether the phase evolution fits into this picture. By choosing a fixed η within the similarity region, eqs. (2.27) and (4.9) imply that for varying t we have

$$\frac{\varphi(0,t) - \varphi(f\eta,t)}{(t_o-t)^{2\beta-1}} = \alpha^2\beta\eta^2/4 = \text{const.} \tag{4.10}$$

By inserting the values obtained numerically for φ, and putting $\beta = 0.61$, we find indeed that the L.H.S. of (4.10) is a constant. The value of this constant yields $\alpha = 1.35$, which agrees very well with the value obtained from the amplitude development.

In the work by Zakharov and Synakh (1975) the wave-field is assumed to have the form

$$u(r,t) = [\tfrac{1}{f}\,\phi(r/f) + \phi_1]\, e^{i\varphi(r,t)} \tag{4.11}$$

where $\varphi(r,t)$ is given by eq. (2.27). They implicitly make the following invalid assumptions:

i) The ordering; $\phi_1 \ll \phi(r/f)/f$ is valid for all r as $t \to t_o$

ii) Eq. (2.27) for φ is valid for all r as $t \to t_o$

iii) ϕ_1 is constant throughout the similarity region.

From these assumptions and eq. (3.4), they arrive at the following asymptotic expression for the energy integral

$$\frac{H}{2\pi} = \int_o^\infty (|u_r|^2 - \tfrac{1}{2}|u|^4)\,r\,dr \sim (f')^2 \int_o^\infty \phi^2\eta^3 d\eta - \frac{2}{f}\,\phi_1 \int_o^\infty \phi^3 \eta\, d\eta \tag{4.12}$$

Conservation of H now implies $f = \alpha(t_o - t)^{2/3}$. We may reconsider this argument by replacing i)-iii) by more reasonable assumptions. Fig. 3 clearly shows that $\phi_1(\eta,t) \to 0$ as $t \to t_o$ for fixed η. One may conjecture that, in the collapse limit, $\phi_1(\eta,t) = 0$ for $\eta < \eta_o(t)$ where η_o is the point where the similarity amplitude $\phi(\eta_o)/f$ is equal to the quasistationary amplitude $u_{III}(r_o)$. This conjecture means that the transition region will vanish in the collapse limit. From the asymptotic form

$$\phi(\eta_o) \sim \left(\frac{\pi}{2\sqrt{\lambda}\ \eta_o}\right)^{\frac{1}{2}} e^{-\sqrt{\lambda}\ \eta_o} \tag{4.13}$$

and assuming $u_{III}(r_o) \to$ const. as $r_o \to 0$, we find

$$e^{-2\sqrt{\lambda}\ \eta_o/f^2} = \text{const.} \times \eta_o \tag{4.14}$$

and

$$\sqrt{\lambda}\ \eta_o = \text{const.} + \ell n(1/f) - \tfrac{1}{2}\ \ell n\ \eta_o \tag{4.15}$$

Hence η_o diverges more slowly than $\ln(1/f)$ as $f(t) \to 0$. By means of (4.13) we find that

$$\int_0^{\eta_o} (\phi_\eta{}^2 - \tfrac{1}{2}\phi^4)\eta d\eta \sim -\frac{\pi}{4\sqrt{\lambda}}\ e^{-2\sqrt{\lambda}\ \eta_o} \quad \text{as } \eta_o \to \infty \tag{4.16}$$

which yields for the energy integral inside η_o (3.4):

$$H(\eta_o) \sim \frac{1}{4}(f')^2 \int_0^{\eta_o} \phi^2\eta^3 d\eta - \frac{\pi}{4\sqrt{\lambda}\ f^2}\ e^{-2\sqrt{\lambda}\ \eta_o} \tag{4.17}$$

By means of (4.15) we have that

$$r_o \equiv f\eta_o < f\ \ell n\ (1/f) \to 0$$

in the collapse limit. We have assumed that $u_{III}(r_o)$ converges to a finite value as $r_o \to 0$, and this means that the energy integral in region III converges to a finite limit

$$2\pi \int_{r_o}^{\infty} (|u_r|^2 - \tfrac{1}{2}|u|^4) r dr = H - H(\eta_o) \to H_{III} \qquad (4.18)$$

Since H is conserved, this implies that $H(\eta_o)$ converges to a finite value in the collapse limit, and from (4.17) we have

$$(f')^2 = \text{const.} \times e^{-2\sqrt{\lambda}\,\eta_o}/f^2 \qquad (4.19)$$

An asymptotic solution to eqs. (4.14) and (4.19) is $\eta_o \sim \ln(1/f)$, $f \sim (t_o-t)\ln(t_o-t)$. The expression for f can also be written as

$$f \sim (t_o-t)^{\beta(t)}, \quad \beta(t) \sim 1 - \tfrac{1}{2}\,\frac{\ln[\ln(t_o-t)^{-1}]}{\ln(t_o-t)^{-1}} \qquad (4.20)$$

Since $\beta(t) \to 1^-$ as $t \to t_o$, $\beta = 1$ would be the most correct exponent in the collapse limit, and the arguments of Zakharov et al. (1971; 1975) for $\beta = 2/3$ do not apply. Also Newell (1978) has presented strong arguments against the $\beta = 2/3$ law using a somewhat different approach.

IV.4 The global, localized similarity solution

In sect. II we found that the solution corresponding to $a = 0$ for $D = 2$ is localized. By substituting the amplitude $|u| = \phi(\eta)/f$ into the expression for the mean square width, we find

$$\langle r^2 \rangle \equiv \frac{2\pi}{N} \int_0^{\infty} |u|^2 r^3 dr = \frac{2\pi f^2}{N} \int_0^{\infty} \phi^2 \eta^3 d\eta = Af^2/(f')^2 \qquad (4.21)$$

Here the constant A can be expressed in terms of the integrals N and H as $A = 4H/N$, since we found in sect. III that

$H = \frac{1}{2}\pi(f')^2 \int_0^\infty \phi^2\eta^3 d\eta$ using (2.23). The general result of virial theory is

$$\langle r^2 \rangle = At^2 + Bt + C \tag{4.22}$$

where A is the constant given above, and

$$B \equiv \partial_t \langle r^2 \rangle_{t=0}, \qquad C \equiv \langle r^2 \rangle_{t=0}$$

A sufficient collapse condition is $A < 0$. However, if $A > 0$, as in this case, collapse still occurs if $-B > 2(AC)^{\frac{1}{2}}$ and the collapse time is $t_o = [|B|-(B^2-4AC)^{\frac{1}{2}}]/2A$. Since the a = 0-solution requires $f(t) = \sqrt{C_1}(t_o-t)$, correspondence between (4.22) and (4.21) is obtained for $-B = 2(AC)^{\frac{1}{2}}$. This means that this solution is exactly on the virial threshold for collapse, and suggests that an infinitesimal perturbation $\delta u(r,0)$ of the initial wave-form may lead either to a final collapse or to dispersion, depending on the nature of the perturbation. The proof of this is given in appendix II.

At this point it is very tempting to suggest that a perturbation, which leads to collapse according to virial theory, will lead to a local collapse development described by the similarity solution (2.30), corresponding to $a > 0$. And, on the other hand, if virial theory predicts avoidance of collapse, the development is described by the non-collapsing similarity solution (2.34), corresponding to $a < 0$.

Indeed, numerical runs confirm this idea. In fig. 6 we show the actual development of $|u(r=0,t)|$ for an initial profile $u(r,t=0) = \tilde{\phi}(r) + \delta\phi(r)$, where $\delta\phi(r)$ is a small positive perturbation and $\tilde{\Phi}(r)$ is a solution to (3.6). This perturbation is easily shown by virial theory to yield avoidance of collapse in the sense that $\langle r^2 \rangle \to \infty$ as $t \to \infty$ without ever becoming negative.

One might conceive that a collapse singularity could develop locally even if $\langle r^2 \rangle$ remains positive definite, however, the numerical solution shows that this does not happen. The dotted curve in fig. 6 shows an almost perfect fit of the non-collapsing similarity solution (2.34) to the actual amplitude development. Thus, the global a = 0-solution is unstable in the sense that any perturbation will lead to a development corresponding to $a \neq 0$, which is qualitatively different from the $a = 0$ development as $t \to t_o$. This concept of stability will be formulated in more precise terms in the next section.

Even though the global $a = 0$ solution is unstable, the arguments of the preceeding paragraph indicate that it reappears in a local form in the later stages of the collapse. The asymptotic form $f(t) \sim (t_o-t)\ln(t_o-t)$ indeed yields $a \to 0^+$ and $\beta \to 1^-$ as $t \to t_o$, and $\phi = \phi(\eta)$ within the spike region. The stability analysis presented above, does not hold for this case, because of the local character of the spike. Thus, the stability of the development of the spike singularity remains uncertain, though the numerical evidence so far indicates a stable behaviour. Certainly, there is a need for more refined numerical schemes implemented on very fast computers in order to achieve more information about the nature of the singularity.

V. PROPER AND IMPROPER SIMILARITY SOLUTIONS

An obvious difficulty in assessing the physical relevance of our similarity solutions is that most of them are non-localized, whereas actual physical wave-forms always are localized. i.e. the wave-function is square-integrable;

$$\int_{\text{all space}} u^* \, u \, d\vec{x} < \infty \tag{5.1}$$

As was demonstrated in the previous section, there is numerical evidence that a wide class of localized initial wave-forms will evolve into a state where they are well approximated by the similarity solution in a finite space-domain. For decaying solutions this proximity is maintained for a semi-infinite time interval (t_1,∞), for collapsing solutions only for a finite interval (t_1,t_2) during the development of the singularity. To this date, there exists no definite proof that any particular similarity solution holds all the way to the instance of collapse, i.e. proximity is maintained only in a time interval (t_1,t_2) where $t_2 < t_o$. This makes it impossible to replace the notion of "local proximity" with the more precise concept of "asymptotic convergence" for collapsing solutions. For decaying solutions this is indeed possible.

Consider a similarity solution $s(\vec{x};t)$ and a localized, evolving wave-form $u(\vec{x};t)$. u is said to be in the ε-neighbourhood of s inside the sphere $r < r_o(t) \equiv f(t)\eta_o$, on the time interval $I_t = (t_1,t_2)$, if

$$\rho_{\eta_o}(u,s;t) \equiv \left[\frac{\int_o^{r_o} |u-s|^2 d\vec{x}}{\int_o^{r_o} |s|^2 d\vec{x}}\right]^{\frac{1}{2}} < \varepsilon \tag{5.2}$$

for all $t \in I_t$. If there exists an $\varepsilon \ll 1$ and an $\eta_o > 0$ such that (5.2) is satisfied, u is locally proximate to s. If I_t is a finite interval, not containing a singularity ($t_1 > t_o$ or $t_2 < t_o$), continuity of u and s ensures that the definition of local proximity is equivalent with the notion of pointwise proximity. This notion results from replacing ρ_{η_o} by its limit

$$\rho_o(u,s;t) = \lim_{\eta_o \to 0} \rho_{\eta_o}(u,s;t) = \frac{|u(\vec{0};t)-s(\vec{0};t)|}{|s(\vec{0};t)|} \tag{5.3}$$

in (5.2). If $t_1 = t_o$ or $t_2 = t_o$, pointwise proximity does not necessarily imply local proximity. However, the pointwise criterion contains the essential information about the time development of the singularity. In this sense one may state that a similarity solution s reveals the nature of the collapsing singularity if

$$\lim_{t \to t_o} \rho_o(u,s;t) = 0 \tag{5.4}$$

For the CSE, no similarity form is known to have this property. For localized similarity solutions a global proximity criterion may be applied by replacing ρ_{η_o} with

$$\rho_\infty(u,s;t) \equiv \lim_{\eta_o \to \infty} \rho_{\eta_o}(u,s;t) \tag{5.5}$$

in (5.2).

For decaying solutions, where the interval $I_t = (t_1, \infty)$ is semi-infinite, local proximity is the relevant notion. Note that in this case $r_o(t) = f(t)\eta_o \to \infty$ as $t \to \infty$.

If $\rho_{\eta_o}(u,s;t)$ is small, and monotonically decreasing on the time interval I_t, u is attracted by s on I_t. If all initial waveforms $v(\vec{x},0)$ in a δ-neighbourhood of $u(\vec{x},0)$ (i.e. all $v(\vec{x},0)$ for which $\int |v(\vec{x},0)-u(\vec{x},0)|^2 d\vec{x} < \delta$) are also attracted by similarity solutions of the same form as s, u is stably attracted by s.

A similarity solution, which stably attracts some localized waveform on some nonzero time interval, is one that may be of physical significance and will be termed a proper similarity solution.

All decaying similarity solutions we have found are proper. In fact, any dispersing waveform (no collapse, no stable solitons) will in the limit $t \to \infty$ be governed by the linear Schrödinger equation. The asymptotic solutions of this equation conforms with our decaying solutions in such a way that $\rho_{\eta_o}(u,s;t\to\infty) = 0$. Our numerical solutions, and the analysis in IV.4 and the appendix show that the localized similarity solution corresponding to $a = 0$, $f(t) = (t_o-t)^{-1}$ is improper for $t < t_o$ due to instability. A completely analogous analysis (though computationally simpler) reveals the same characteristics for the 2-dimensional soliton; it will collapse or disperse depending on the nature of an infinitesimal perturbation (see Sec. VI). An interesting class of improper solutions are given by eq. (2.40) for $C_3 < 0$, $C_4 > 0$ and $t < t_o \equiv |C_4/C_3|$. These solutions are non-localized, and they "blow up" as $t \to t_o$. Consider a localized wave-packet which is locally proximate to such a solution for $\eta < \eta_o$. If this proximity was to persist as $t \to t_o$ for D = 1 this would imply a 1-dimensional collapse. However, from eq. (3.1) we find that such a proximity would imply that $N \geq f^{-1}R\phi_o^2\eta_o \to \infty$ as $t \to t_o$, i.e. the conservation of N would be violated. Hence, no localized wave could be proximate to this similarity solution on an interval (t,t_o), and the similarity solution must be improper.

VI SOLITON SOLUTIONS

In this section we consider the stationary solutions of the CSE eq. (2.2) vanishing at infinity. Here we shall call such solutions for solitons even if they turn out to be unstable. We seek the stationary solutions in the form

$$u(\vec{x},t) = \Phi(\vec{x})e^{i\lambda^2 t} \tag{6.1}$$

which on substitution into eq. 2.2 yields the following equation for $\Phi(\vec{x})$

$$\nabla^2\Phi + \Phi^3 - \lambda^2\Phi = 0 \tag{6.2}$$

where Φ is real.

Note that the transformation of eq. 2.2 into (6.2) formally can be obtained from the similarity transformation (2.3-2.4) with $f(t) = h(t) =$ const. or from the linear group transformation with k arbitrary and $\ell = m = 0$ (see eq. 2.56).

Before describing the solutions of eq. 6.2 in radial symmetry for one -, two- and three dimensions we shall discuss some basic features of the solitons and in particular we consider their stability. The CSE (2.2) belongs to the Hamiltonian type (e.g. Zakharov et al. 1983) and may be written:

$$iu_t = \frac{\delta H}{\delta u^*} \tag{6.3}$$

where the star denotes the complex conjugate, H is the Hamiltonian:

$$H \equiv \int \kappa d^D r = \int (|\nabla u|^2 - \tfrac{1}{2}|u|^4) d\vec{x} \tag{6.4}$$

and δ denotes variation in the usual way i.e.

$$\frac{\delta H}{\delta u^*} \equiv \frac{\partial \kappa}{\partial u^*} - \partial_\ell \frac{\partial \kappa}{\partial(\partial_\ell u^*)} \quad .$$

Except for H eq. 2.2 also conserves the plasmon number

$$N = \int |u|^2 \, d\vec{x} \tag{6.5}$$

and the momentum

$$\vec{P} = \frac{i}{2} \int (u^* \nabla u - u \nabla u^*) d\vec{x} \tag{6.6}$$

see also Sec. III where these quantities are given for the radial symmetric case, here $\vec{P} \equiv \vec{0}$ as it is for a solution of the type (6.1). The parameter λ^2 in eq. 6.2, which denotes the nonlinear frequency shift, can be removed by a simple scaling of the variables (see also sec. III) introducing

$$\tilde{\Phi} = \Phi/\lambda \text{ and } \tilde{\vec{x}} \equiv \vec{x}\lambda \tag{6.7}$$

eq. (6.2) reads:

$$\nabla^2 \tilde{\Phi} + \tilde{\Phi}^3 - \tilde{\Phi} = 0 \tag{6.8}$$

Thus a solution of (6.8) immediately gives us the solution of (6.2) by using (6.7) showing that the soliton amplitude is proportional to λ while the width is inversely proportional to λ. The conserved quantities N and H for solutions of the form (6.1) reads:

$$N = \int \Phi^2 \, d\vec{x} = \lambda^{2-D} \int (\tilde{\Phi})^2 \, d\vec{x} \equiv \lambda^{2-D} \tilde{N} \tag{6.9}$$

$$\begin{aligned} H &= \int (|\nabla\Phi|^2 - \tfrac{1}{2}\Phi^4) d\vec{x} \\ &= \lambda^{4-D} \int (|\nabla\tilde{\Phi}|^2 - \tfrac{1}{2}(\tilde{\Phi})^4) d\vec{x} \\ &\equiv \lambda^{4-D} \tilde{H} \end{aligned} \tag{6.10}$$

where $\tilde{N}$ and $\tilde{H}$ are the plasmon number and Hamiltonian for the solution of 6.8, respectively.

From the formulation of the CSE in the Hamiltonian form it follows that the soliton solutions of the form (6.1) can be found from the variational problem

$$\delta(H + \lambda^2 N) = 0 \tag{6.11}$$

and the solitons are extremum points of H for fixed N (Zakharov et al. 1983). For the soliton solution a relation between H and N can be obtained by following the procedure of Zakharov et al. (1983). First we multiply the equation

$$\frac{\delta}{\delta\Phi}(H + \lambda^2 N) = 0$$

by Φ and integrate over $\vec{x}$ resulting in

$$I_1 - 2I_4 + \lambda^2 N = 0 \tag{6.12}$$

where we have introduced:

$$I_1 = \int |\nabla\Phi|^2 d\vec{x} \quad \text{and} \tag{6.13a}$$

$$I_4 = \frac{1}{2}\int \Phi^4 d\vec{x} \tag{6.13b}$$

i.e. $H = I_1 - I_4$. A second relation between I_1, I_4 and N is obtained by considering a scaling transformation (Derrick 1964)

$$\Phi = \alpha^{D/2}\, \Phi_o(\alpha\vec{x}) \tag{6.14}$$

which leaves N invariant, whence

$$H = \alpha^2 I_1 - \alpha^D I_4 \quad ; \tag{6.15}$$

since Φ_o is a solution of (6.11) and N is independent of α we must have

$$\frac{\partial}{\partial\alpha} H(\alpha) \Big|_{\alpha=1} = 0$$

resulting in

$$I_1 = \frac{D}{2} I_4 \tag{6.16}$$

and

$$\frac{\partial^2 H}{\partial\alpha^2}\Big|_{\alpha=1} = D(2-D)I_4 \quad . \tag{6.17}$$

Combining (6.12) and (6.16) results in an relation between H and N:

$$H = \frac{D-2}{4-D}\lambda^2 N \tag{6.18}$$

Note that there is a misprint in the corresponding relation in the report of Zakharov et al. (1983).

The solitons are realized for extremum points of H with fixed N. We see from (6.14) that only for D < 2 is the Hamiltonian bounded from below and the soliton solution corresponds to a minimum of H for fixed N. Thus only for $D < 2$ is the soliton absolutely stable (e.g. Derrick 1964 and Zakharov et al. 1983). In two dimensions H and its derivatives are zero and the soliton is marginally stable, while the three-dimensional soliton is unstable and corresponds to a maximum of H.

Here we shall further consider the soliton stability using different methods, and in particular we shall discuss the applicability of the virial theorem (see Sec. I) to predict stability. A particular simple criterion for longitudinal stability of one-dimensional soliton solutions to Schrödinger-type equations is the socalled N-theorem, which was first derived by Kolokolov (1974) using the Lyapunov method and later generalized by Laedke et al. (1983). The condition, which is sufficient and necessary for longitudinal stability reads

$$\partial_{\lambda^2} N > 0 \tag{6.19}$$

where λ^2 is the nonlinear frequency shift. Laedke and Spatchek (1984) generalized (6.19) to higher dimensions and vector-fields. They used it to prove that a nonlinear Schrödinger equation with exponential nonlinearity (i.e. the term $|u|^2$ in (2.2) is replaced by $1-\exp(-|u|^2)$) has stable three-dimensional spherical soliton solutions when the amplitude is sufficiently large. In addition to the longitudinal stability (i.e. stability with respect to r-dependent perturbations) Laedke and Spatchek showed that this spherical solition is completely stable with respect to all types of perturbations. By using a similar method Vakhilov and Kolokolov

(1973) showed that a cylindrical symmetric soliton solution to the nonlinear Schrödinger equation with saturable non-linearity (i.e. $|u|^2$ is replaced by $|u|^2/(1+|u|^2)$) is stable.

Applying (6.19) to the soliton solutions (6.1) of (2.2) we obtain by using (6.9):

$$\partial_{\lambda} 2N = \frac{1}{2} (2-D)\lambda^{-D} \tilde{N} \quad . \tag{6.20}$$

The stability criterion obtained from (6.20) $D < 2$ coincides with the condition for the boundedness of the Hamiltonian. Thus the one-dimensional soliton is longitudinal stable, the two-dimensional soliton is marginally stable, while the three-dimensional soliton is unstable. We re-emphasize that (6.19) is only concerned with longitudinal stability, and it is well-known that the one-dimensional soliton is unstable with respect to transverse perturbations (e.g. Zakharov and Rubenchik 1973, Zakharov et al. 1983), an instability which, at least to third order, leads to unlimited growth (Janssen and Rasmussen 1983). Also the cylindrical symmetric two-dimensional soliton is unstable with respect to a perturbation along the z-direction (e.g. Zakharov and Rubenchik 1973, Han 1979, Zakharov et al. 1983).

The virial theory, which is briefly described in Sec. I, gives a <u>sufficient</u> criterion for collapse i.e. instability of localized solutions to the CSE. For solutions of the form (6.1) the virial theory results in:

$$\partial^2_t < (\Delta \vec{x})^2 > = \frac{2}{N} V \equiv \frac{2}{N} \left[4H + (2-D)\int \Phi^4 d\vec{x}\right] \quad . \tag{6.21}$$

From (6.21) it is seen that $H < 0$ is a sufficient criterion for collapse only for two and three dimensions since the last term for $D = 1$ is always positive. That $H < 0$ is not a necessary criterion for collapse, as erroneously stated in the last paragraph in the paper by Laedke and Spatchek (1984) is easily seen: For $D = 2$ (6.21) may be integrated to yield

$$\langle(\Delta\vec{x})^2\rangle = \frac{4H}{N}t^2 + Bt + C \qquad (6.22)$$

thus even if $H > 0$ collapse may still take place for $-B > 2(\frac{4H}{N}C)^{1/2}$ (see also Sec. IV.4). For $D = 3$ the last term in (6.18) is always negative and V can easily be negative for $H > 0$, which will lead to collapse. Therefore we believe that the arguments given in the last paragraph of the paper by Laedke and Spatchek (1984), that a virial type of collapse is not to be expected for the spherical soliton solution of the CSE, are incorrect.

Applying the virial theorem (6.21) to the soliton solution we find

$$V \equiv 4H + (2-D)2I_4 \equiv 0 \qquad (6.23)$$

for any D by using the relation (6.16). Thus the soliton solutions are all on the "virial theory" threshold for collapse. However, by using qualitative arguments we may consider the stability and the evolution of the unstable solitions. Imagine that the soliton solution is perturbed initially:

$$u(\vec{x},0) = \Phi_S(\vec{x}) + \delta u(\vec{x}) \quad . \tag{6.24}$$

where Φ_S is the soliton solution of (6.2). For the perturbed soliton V takes the form

$$V = 4\delta H + (2-D)2\delta I_4 \tag{6.25}$$

where δH and δI_4 are the perturbations of H and I_4 due to δu. In (6.25) δH is conserved during the evolution of the soliton. For D = 2 the virial theorem is written as (6.22) where $B = \partial_t < (\Delta\vec{x})^2 > |_{t=0}$ and $C = < (\Delta\vec{x})^2 > |_{t=0}$ and the arguments are rigorous. For the soliton solution B = 0. If $\delta H < 0$ the perturbed soliton collapses. If $\delta H > 0$ collapse takes place for $-\delta B > 2 \left(\frac{4\delta H}{N} C\right)^{1/2}$ while the soliton spreads for the opposite case. That there excists perturbations δu giving rise to these perturbations of H and B may be proved by using the argumentation of Appendix II. For D = 1 and 3 the term with δI_4 which is not conserved appears in (6.25). An estimate of I_4 can be given by using that N is conserved which yields

$$|u|^2 \propto \frac{N}{L^D}$$

where L is some averaged width of the perturbed soliton $L = \sqrt{<(\Delta x)^2>}$. Thus

$$I_4 = \frac{1}{2}\int |u|^4 d\vec{x} \propto \frac{N^2}{L^D}$$

which means that for a collapsing solution I_4 increases while for a spreading soliton I_4 decreases. Consider first $D = 1$, if $V < 0$ then $\langle(\Delta x)^2\rangle$ starts decreasing implying an increase of δI_4 and in turn of V (δH is conserved), when $V = 0$ the evolution stops. For $V > 0$, $\langle(\Delta x)^2\rangle$ starts increasing implying a decrease of δI_4 and in turn of V until the evolution stops at $V = 0$. For $D = 3$ the sign in front of δI_4 is negative, and $V = 0$ will always lead to collapse while $V > 0$ may result in a spreading of the soliton. Similar arguments for the evolution of Gaussian wave packets in one, two and three dimensions were presented by Goldman et al. (1980). The foregoing arguments based on the "virial theory" confirm that only the one-dimensional soliton is absolutely stable, while the perturbed soliton solution in two and three dimensions either collapses or disperses spatially depending on the "sign" of the perturbation

Let us finally consider the soliton solutions of Eq. (6.8) in radial symmetry:

$$\frac{d^2\Phi}{dr^2} + \frac{D-1}{r}\frac{d\Phi}{dr} + \Phi^3 - \Phi = 0 \qquad (6.26)$$

where the tilde ~ has been dropped. The soliton solutions satisfy the boundary conditions:

$$\left.\frac{d\Phi}{dr}\right|_{r=0} = 0 \text{ and } \Phi \to 0 \text{ for } r \to \infty \quad . \qquad (6.27)$$

Solutions of (6.2) are easily obtained by using the scaling (6.7). For D = 1 eq. (6.26) is solved by quadrature, and it has one and only one solution satisfying Eq. (6.27) namely the well-known:

$$\Phi = \sqrt{2}\ \text{sech}\ r \tag{6.28}$$

In two and three dimensions (6.26) cannot be solved analytically. For D = 2 (6.26) was solved numerically by Chiao et al. (1964). For this case (6.26) have a denumerable set of solutions (e.g. Yankauskas 1966) satisfying (6.27) and the j^{th} solution has j zeroes. The j = 0 solution is monotonically decreasing and only this solution was found by Chiao et al. (1964). The results for D = 2 can easily be generalized to D = 3. The zeroth order solutions for D = 2 and 3 are shown in Fig. 7 together with the one-dimensional soliton eq. (6.28). In table 1 we show the values of $\Phi(r = 0)$, N and H. For D = 2 and 3 N and H are calculated numerically by using eqs. (6.9 - 6.10) in radial symmetry (i.e. eqs. (3.1 - 3.2) with 6.1 inserted) and for D = 1 they are calculated analytically. The relation between N and H is in perfect agreement with eq. 6.15. Finally, also V (eq. 6.21) was calculated (for D = 2 and 3 numerically), and it was found to be equal to zero for all cases as predicted by (6.23).

D	$\Phi(0)$	N	H
1	$\sqrt{2}$	4	$-\frac{4}{3}$
2	2.2062	11.68	0
3	4.3374	18.94	18.94

Table 1. *Parameters for the soliton solutions of* Eq. (6.26) *or* (6.2) with $\lambda = 1$. The parameters for $\lambda \neq 1$ can be obtained from (6.7), (6.9) and (6.10).

VII CONCLUSIONS

We have presented a detailed study of the similarity reductions of the Cubic Schrödinger Equation, with special emphasis on the radially symmetric cases. For two and three dimensions we found that the infinitisimal transformation group of Lie results in the same similarity solutions as the "general" transformation (2.3-2.4). The two dimensional case was found to be a special case allowing for further analytical treatment and a number of similarity solutions, which are specific for this dimensionality, were found. Furthermore, we showed that localized similarity solutions are possible only in two dimensions. By a localized similarity solution we mean a solution for which $\Phi(\eta) \to 0$ as $\eta \to \infty$ and the conserved quantities of the CSE are finite (Sec. III).

Our main aim was to investigate the physical relevance of the similarity solutions to describe wave collapse. For that purpose we performed a detailed numerical study of the collapsing solutions of the CSE in two dimensional radial symmetry. Our main conclusions are as follows:

i) The similarity behaviour is local in space and time. Thus, the question as to whether the similarity solution is localized or not is normally irrelevant for the actual collapse evolution, and the evolution of a wide class of localized wave-forms are locally attracted to non-localized similarity solutions. Furthermore, a specific time dependence of the collapsing spike maximum, say e.g. $|u| \propto (t_0-t)^{-\beta}$ with $\beta = \frac{1}{2}$, *holds only in a finite* time regime, with $t_2 < t_0$. Closer to collapse β increases and we have argued that $\beta \to 1^-$ as $t \to t_0$.

ii) The localized similarity solution for which $|u| \propto (t_0-t)^{-1}$ is shown to be unstable. This is, perturbations of the initial wave-form will alter the collapse behaviour significantly or even prevent the collapse. By introducing the concepts of proper and improper similarity solutions in Sec. V, we concluded that this localized similarity solution is the only improper solution we have considered. Therefore, the existence of localized similarity solutions is neither a sufficient nor a necessary criterion for self-similar collapse.

iii) In our investigations the similarity reduction is performed by eliminating the time variable in the CSE. If, however, a slow explicit time dependence is permitted in the reduced equations, a new class of quasi-similarity solutions arises. We have numerical and analytical evidence that the explicit slow time dependence disappears as the collapse is approached.

Finally, we have also investigated the stationary or soliton solutions of the CSE since these appear for a particular similarity reduction (f(t) = h(t) = const. in eqs. 2.3-2,4). We have reviewed different stability investigations of these solitons and found that only in one dimension is the solition absolutely stable.

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Appendix I

The self-similar solution with constant amplitude $\Phi(\vec{\eta}) = \Phi_o$. Inserting (2.36) and (2.37) into (2.11) results in

$$g(\vec{\eta}) = \frac{\eta^2}{D}\left[\frac{(ff')^2}{2}\left(1-\frac{2}{D}\right) - \frac{f^3f''}{2}\right]$$

$$+ \vec{C}_1\cdot\vec{\eta}\left[\frac{(ff')^2}{2}\left(1-\frac{3}{D}\right) - \frac{f^3f''}{2D}\right] \qquad (I.1)$$

$$- C_2\left[\frac{(ff')^2}{2} + \frac{f^3f''}{2}\right] - C_1^2\frac{(ff')^2}{4} - G'(t)f^2 = -\Phi_o \quad .$$

A non-trivial solution for f(t) (i.e. f(t) ≠ const., which would only give the nonlinear frequency shift proportional to Φ_o^2) must satisfy:

$$(ff')^2\ (D-2) - Df^3f'' = 0 \qquad (a)$$

$$\vec{C}_1[(ff')^2(D-3) - f^3f''] = \vec{0} \qquad (b)\ (I.2)$$

$$\tfrac{1}{2}C_2[(ff')^2 + f^3f''] + \tfrac{1}{4}C_1^2(ff')^2 + G'(t)f^2 = \Phi_o^2 \qquad (c)$$

(a) and (b) can only be satisfied simultaneously for $\vec{C} \equiv \vec{0}$ (since D must be an integer), then f (t) is found from (a):

$$\left(\frac{2-D}{D}\right)(f')^2 + ff'' = 0 \qquad (I.3)$$

which yields

$$f(t) = \left[2(C_3 t + C_4)/D\right]^{D/2} \quad . \tag{I.4}$$

The phase φ is found from eq. (2.8)

$$\varphi = \frac{1}{D} \; \frac{ff'}{2}\eta^2 + C_2 \; \frac{ff'}{2} + G(t) \tag{I.5}$$

we may here include the t-dependent term $C_2 \, \frac{ff'}{2}$ in G(t) and then this new G(t) is determined from (I.2c):

$$G'(t) f^2 = \Phi_o^2$$

resulting in:

$$G\,(t) = \frac{\Phi_o^2}{2C_3} \,\ln\,(C_3 t + C_4) + C_5 \quad \text{for } D = 1$$

and

$$G(t) = \frac{\Phi_o^2 (D/2)^D}{C_3 (D-1)} \quad \left[C_3 t + C_4\right]^{1-D} + C_5 \text{ for } D \neq 1 \ . \tag{I.6}$$

Appendix II

Assume that the unperturbed initial wave-profile $u_o(r)$ corresponds to the global, localized similarity solution, i.e. the situation is marginally stable with respect to collapse/non-collapse in the sense that the unperturbed coefficients A_o, B_o, C_o, as defined in Sec. IV.4, satisfy the relation

$$-B_o = 2(A_oC_o)^{1/2} \quad . \tag{II.1}$$

An infinitesimal perturbation of the initial profile, $u(r,o) = u_o(r) + \delta u(r)$, will lead to collapse if the collapse criterion

$$-B > 2(AC)^{1/2} \tag{II.2}$$

holds for the perturbed quantities $A = A_o + \delta A$ etc. To the lowest order in the perturbations the collapse criterion (II.2) can be written in the form

$$L\{u\} \equiv t_o^2 A + t_o B + C < 0 \tag{II.3}$$

where $t_o = - B_o/2A_o$. From Eq. (II.1) one observes that the functional $L\{u\}$ has the property

$$L\{u_o\} \equiv t_o^2 A_o + t_o B_o + C_o = 0 \quad . \tag{II.4}$$

Let us write the perturbed field in the form $\delta u_\varepsilon(r) = \varepsilon v(r)$ where $v(r)$ is some arbitrary square-integrable function. For $\varepsilon \to 0^\pm$, and a given $v(r)$, the collapse criterion is

$$\lim_{\varepsilon\to 0^\pm} \operatorname{sgn} L\{u_\varepsilon\} = \lim_{\varepsilon\to 0^\pm} \operatorname{sgn}\left(\varepsilon \frac{dL}{d\varepsilon}\right) = -1 \quad . \tag{II.5}$$

Thus, if $(dL/d\varepsilon)_{\varepsilon=0} \neq 0$, the situation shifts from collapsing to non-collapsing when ε changes sign, i.e. the wave-packet may be brought to collapse or disperse as $t \to \infty$, depending on the form of the initial perturbation.

However, the proof is not complete until we have shown that $(dL/d\varepsilon)_{\varepsilon=0} \neq 0$ at least for one choice of $v(r)$. If this was not true, the variational property

$$\delta L\{u\} = 0 \tag{II.6}$$

would hold true for $u = u_o$. By writing the initial field on the form

$$u(r,0) = \frac{1}{t_o} \phi(\eta) \exp\left[i\frac{t_o}{2}\phi(\eta)\right], \eta \equiv r/t_o \tag{II.7}$$

we find that

$$C \equiv \langle r^2\rangle_{t=0} = \frac{2\pi}{N} t_o^2 \int_o^\infty \eta^3 d\eta \phi^2 \tag{II.8}$$

$$B \equiv \partial_t\langle r^2\rangle_{t=0} = \int r^2 d\vec{r} \partial_t^2 |u|^2 = \int i r^2 dr \nabla\cdot(u^*\nabla u - u\nabla u^*)$$

$$= \frac{2\pi}{N} \frac{2}{t_o} \int_o^\infty r^2 dr \phi^2 \partial_r \phi = \frac{2\pi}{N} 2t_o \int_o^\infty \phi^2 \eta^2 \theta^2 d\eta \tag{II.9}$$

where $\theta \equiv \frac{d\phi}{d\eta}$. Furthermore, we have

$$A \equiv \frac{4H}{N} = \frac{2\pi}{N}\left[\frac{4}{t_0^2}\int_0^\infty \eta d\eta\left[\left(\frac{d\phi}{d\eta}\right)^2 - \frac{1}{2}\phi^4\right] + \int_0^\infty \eta d\eta \phi^2\theta^2\right] \quad \text{(II.10)}$$

and by inserting eqs. (II.8-10) into (II.3), we get

$$L\{\phi,\theta\} = \frac{2\pi}{N}\int_0^\infty \eta d\eta\{4[(\frac{d\phi}{d\eta})^2 - \frac{1}{2}\phi^4] + t_0^2\phi^2(\theta + \eta)^2\}. \quad \text{(II.11)}$$

Here the real functions $\phi(\eta),\theta(\eta)$ have been used to represent the complex function $u(r,0)$. The Euler-Lagrange equations corresponding to the variational principle (II.6) are

$$\frac{d}{d\eta}\left(\eta \frac{d\phi}{d\eta}\right) + \eta\phi^3 - t_0^2\eta\phi(\theta + \eta)^2 = 0 \quad \text{(II.12)}$$

$$\phi^2(\theta + \eta) = 0 \quad . \quad \text{(II.13)}$$

Apart from the trivial solution $\phi \equiv 0$, this yields $\theta = -\eta$, and eq. (II.12) reduces to eq. (2.28) with $a = \lambda = 0$. But for $a = 0$, eq. (2.28) exhibits localized solutions only if $\lambda > 0$. Hence, if $\phi(\eta)$ corresponds to the localized similarity solution (i.e it satifies eq. (2.28) for $a = 0$ and $\lambda > 0$), eq. (II.12) cannot be satisfied. This proves that (II.6) does not hold true for $u = u_0$, and thus that $(dL/d\varepsilon)_{\varepsilon=0} \neq 0$ for at least one $v(r)$.

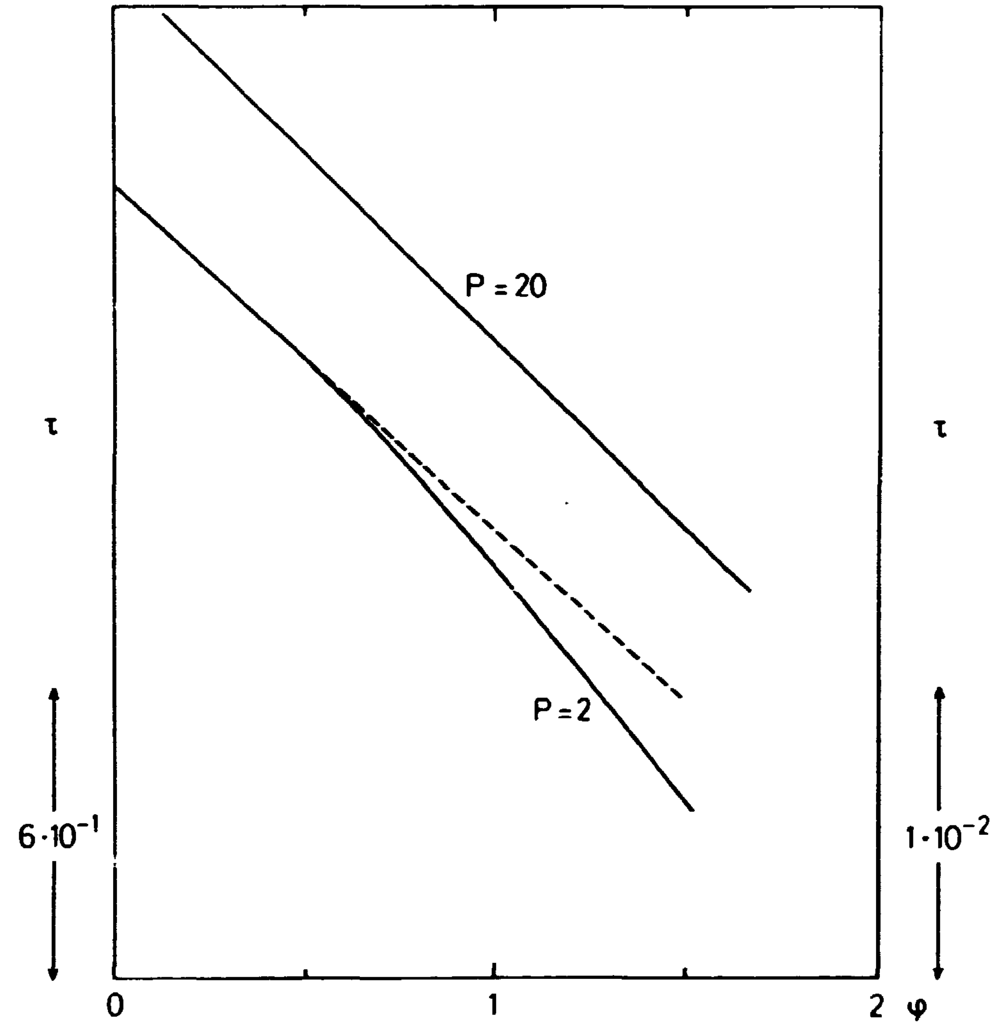
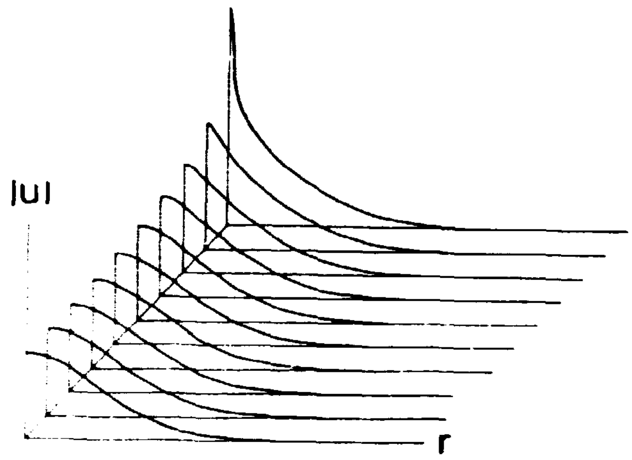
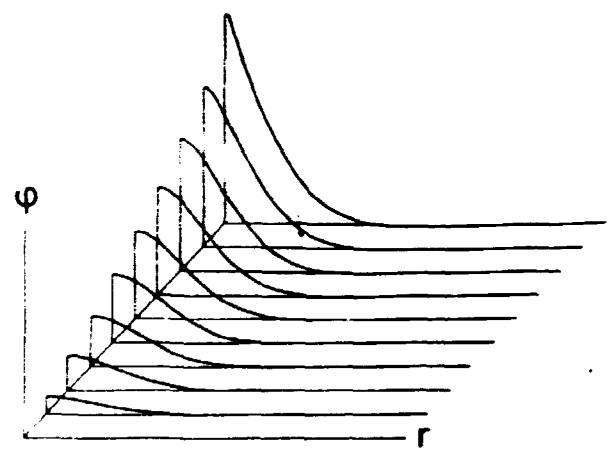


Fig. 1. Temporal evolution of the phase $\varphi(0,t)$ of the collapsing wave-packetfor $P = 2$ and $P = 20$, $\tau \equiv t_0 - t$.



2 a



2 b

Fig. 2. Space-time evolution of the collapsing wave-packet for $P = 100$. a) Evolution of the amplitude $|u|$. b) Evolution of the phase φ.

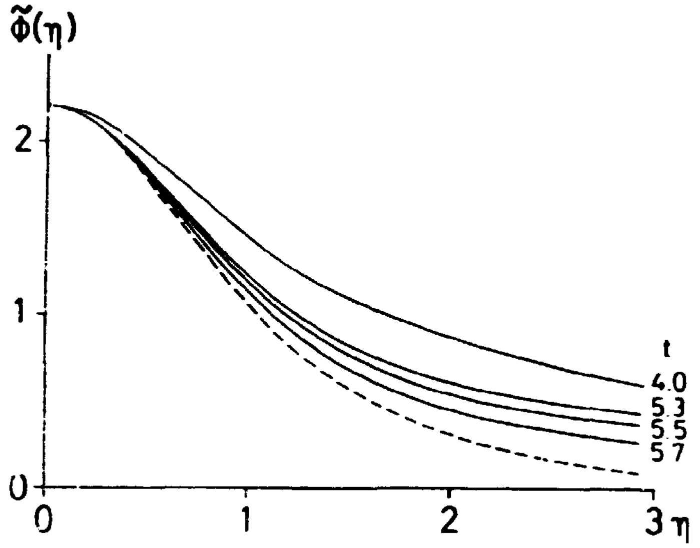
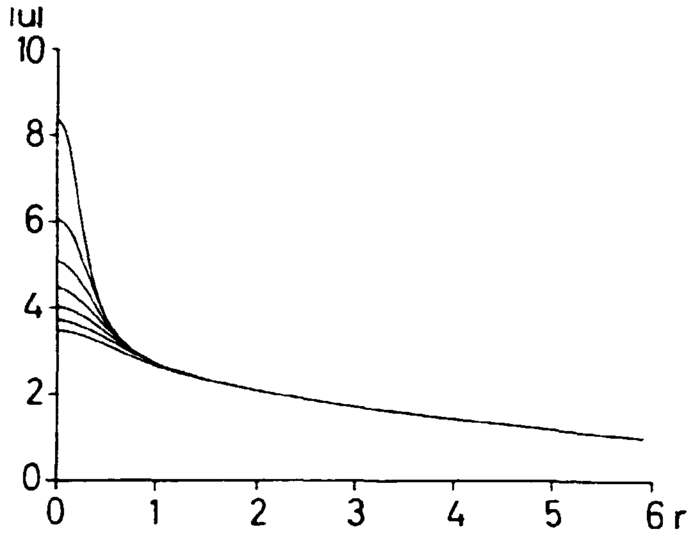
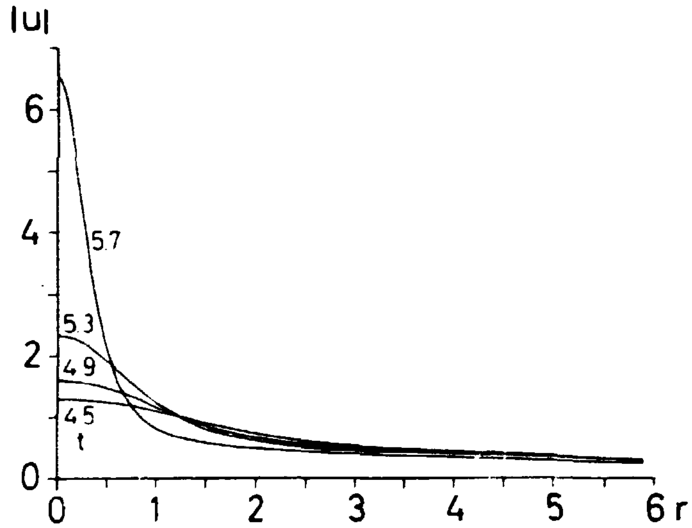


Fig. 3. Comparison of the amplitude profile $|u(r,t)|$ of the collapsing wave-packet with the self-similar solution (P = 2). We have plotted $\tilde{\Phi}(\eta;t) \equiv f|u(\eta f,t)|$ as described in connection with eq. 4.3 The broken curve shows the localized self-similar solutions (the solution of eq. 2.28 with a = 0).



4 a



4 b

Fig. 4. Detailed behaviour of the collapsing spike. a) Earlier stage of collapse in the $\beta = \frac{1}{2}$-regime (P = 20). b) Closer to collapse $\beta > \frac{1}{2}$ (P = 2, $\beta \approx 0.61$).

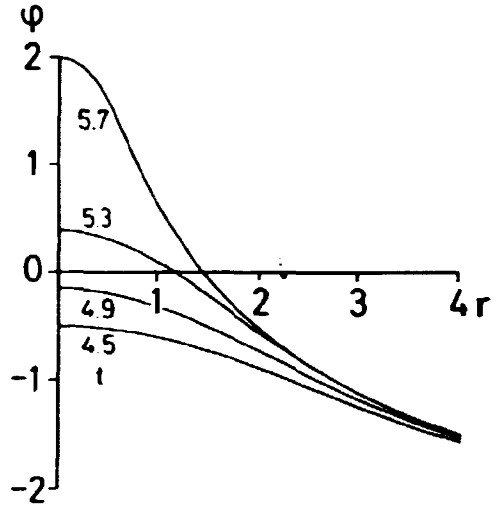


Fig. 5. Spatial structure of the phase φ at various times (P = 2).

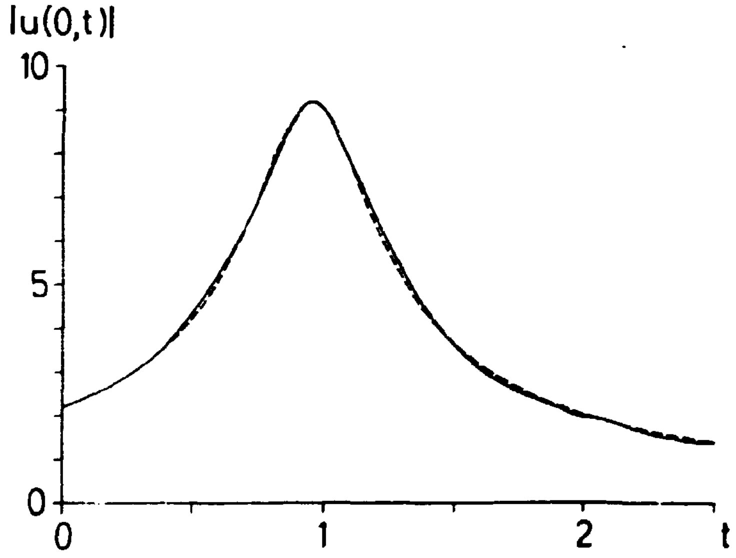


Fig. 6. Development of $|u(r = 0,t)|$ for an initial profile $u(r,t = 0) = \tilde{\Phi}(r) + \delta\Phi(r)$, where $\delta\Phi(r)$ is a small positive perturbation and $\tilde{\Phi}(r)$ is the solution of eq. 2.28 with $a = 0$. The broken line shows the time evolution of the non-collapsing similarity solution ($a < 0$) eq. 2.34.

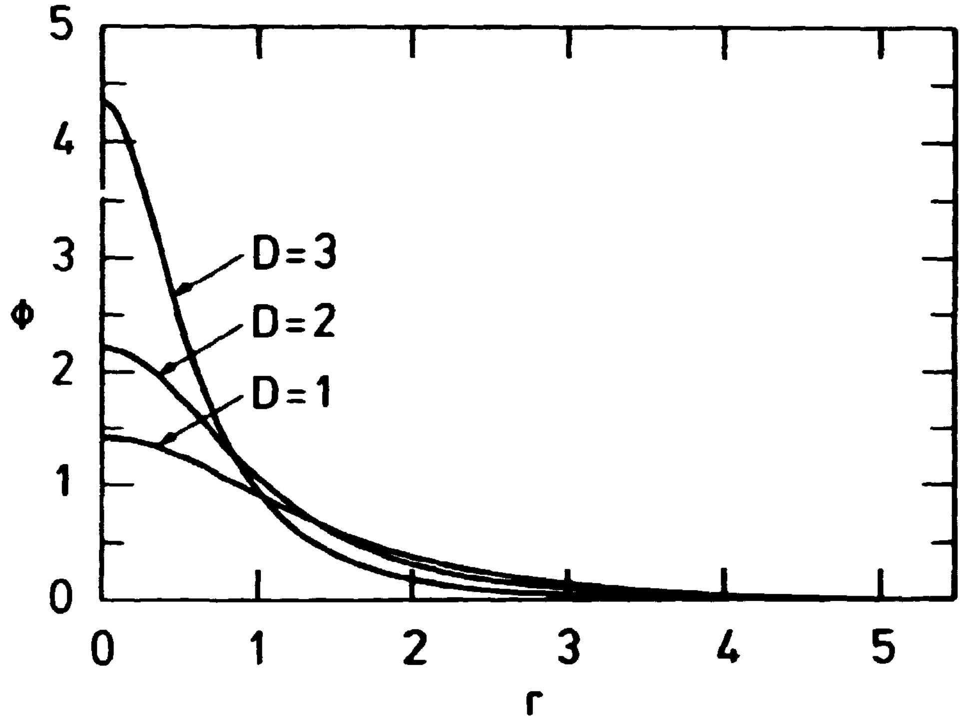


Fig. 7. Soliton solution of the CSE in 1, 2 and 3 dimensions.

Risø National Laboratory Risø - M - 2448

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Title and author(s)

Similarity Solutions of the Cubic Schrödinger Equation and their Role in the Development of Wave-Collapse

Kristoffer Rypdal, IMR, Universiry of Tromsø
Jens Juul Rasmussen, Knud Thomsen

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Abstract

Similarity transformations of the Cubic Schrödinger Equation (CSE) are investeigated. The similarity transformations are used to remove explicit time variation in the CSE and transform it into differential equations in the spatial vatiables only. Different methods for similarity reduction are employed and compared. The main purpose of this investigation is to study the significance of the similarity solutions in the evolution of a collapsing wave packet. Numerical solutions of the CSE in radial symmetry demonstrate that the similarity behaviour is local in space and time, and the similarity solutions are classified by invoking the concept of proper and improper solutions. The nature of the collapsing singulatiry is reexamined and finally, soliton solutions to the CSE are considered.

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